Grading Structure for Derivations of Group Algebras

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Derivations

Definition

A derivation over a group algebra $\mathbb{C}[G]$ is a linear operator d that satisfies the Leibniz rule

$$d(ab) = d(a)b + ad(b) \tag{1}$$

Derivations form an algebra $Der(\mathbb{C}[G])$

Main examples

- Inner derivations: $d_a(x) = [x, a]$ for $a \in G$; they form an ideal $InnDer(\mathbb{C}[G])$.
- Central derivations: $d_{\tau,z}(x) = \tau(x)xz$ for $x \in G, z \in Z(G), \tau : G \to (\mathbb{C}, +)$ — a homomorphism; they form a subalgebra $ZDer(\mathbb{C}[G])$.

Main Result

The focus of the talk is describing a **nontrivial** grading for $Der(\mathbb{C}[G])$. More formally:

Theorem

If $|G/G'| > 1$, $Der(\mathbb{C}[G])$ is ***nontrivially*** *graded with G/G', that is there exist*[a] *such Der_k, $k \in G/G'$, that*

$$Der(\mathbb{C}[G]) = \bigoplus_{k \in G/G'} Der_k \tag{2}$$

$$[Der_k, Der_l] \subset Der_{kl}$$

[a] arXiv:2308.00512

Note

Generally, nontriviality would mean the existence of non-neutral element k such that $Der_k \neq 0$. However, we will show that $Der_k \neq 0$ **for all** k.

Definition of Der_k needs further definitions.

Groupoid $\Gamma(G)$

Definition

For a given group G consider a **small groupoid** $\Gamma(G)$:

1. **objects** (Obj) — elements of G,
2. **arrows** (Hom) — pairs of elements of G. For an arrow (u,v) its source $S(u,v) := v^{-1}u$, and its target $T(u,v) := uv^{-1}$. $Hom(a,b)$ denotes a set of all arrows for which the source is a and target is b.
3. Consider two arrows $(u_2,v_2) \in Hom(b,c), (u_1,v_1) \in Hom(a,b)$. The **composition** for these two arrows is given by:

$$(u_2,v_2) \circ (u_1,v_1) := (v_2u_1, v_2v_1) \in Hom(a,c) \tag{3}$$

Composition

Decomposition

For $g \in G$ denote $[x]$ as a corresponding conjugacy class

$$[x] := \{gxg^{-1} | g \in G\}. \tag{4}$$

G^G denotes the set of conjugacy classes.

It is easy to see that $Hom(a, b)$ is non-empty iff $[a] = [b]$. This yields the following definition and decomposition:

Definition

Let $\Gamma(G)_{[a]}$ denote $\Gamma(G)$'s restriction to objects from $[a] \subset G$.

Proposition

$\Gamma(G) = \bigsqcup_{[a] \in G^G} \Gamma_{[a]}$

Macrodecomposition

Recall an elementary group-theoretical exercise:

Statement

$[a] \subset aG'$ for every $a \in G$

This observation justifies the following symbol (which we will need later)

Definition

$\Gamma_k := \bigcup_{a\in k} \Gamma_{[a]}$, for $k \in G/G'$.

Note that $k \in G/G'$ is a coset.
Now we have a macrodecomposition

$$\Gamma(G) = \bigsqcup_{k\in G/G'} \Gamma_k \tag{5}$$

Characters

The following definition is crucial for the construction of the grading.

Definition

A map $\chi : \textbf{Hom}\ (\Gamma(G)) \to \mathbb{C}$ is a (**locally-finite**) character iff for all composable pairs $(u_1, v_1), (u_2, v_2)$ holds

$$\chi((u_1, v_1) \circ (u_2, v_2)) = \chi(u_1, v1) + \chi(u_2, v2). \tag{6}$$

and for all x holds:

$$\chi(x, y) \neq 0 \text{ for just finitely many } y \tag{7}$$

Definition

The support of a character χ ($supp\chi$) is a set of characters on which χ does **not** vanish:

$$supp\chi := \{(u, v) \in Hom(\Gamma) | \chi(u, v) \neq 0\} \tag{8}$$

The following theorem links characters and derivations and, thus, motivates former.

Theorem

A linear map $d = d(x) = \sum_{h \in G} d_h^x h$ is a derivation iff[a] *there exists a character χ such that*

$$\chi(x, h) = d_h^x \tag{9}$$

[a] A. A. Arutyunov, A. S. Mishchenko, A. I. Shtern, Derivations of group algebras, Fundam. Prikl. Mat., 21:6 (2016), 65-78

Informally, this means that characters are "matrices"of derivations.

Examples

- For an inner derivation $d_a = [x, a]$ $(a \in G)$ the character is

$$\chi_a(u,v) = \begin{cases} 1, & a = S(u,v), \\ -1, & a = T(u,v), \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

- For a central derivation $d_{\tau,z}$ the character is:

$$\chi_{\tau,z}(u,v) = \begin{cases} \tau(uv^{-1}), & v = z, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

It follows that central derivations are non inner because they are not trivial on loops in contrast to inner.

What are Der_k?

Definition

$Der_k = \{d \in Der(\mathbb{C}[G]) : supp\chi_d \subset \Gamma_k\}$

Recall that $k \in G/G'$ is a coset.
The main result can be stated fully now.

Theorem

If $|G/G'| > 1$, $Der(\mathbb{C}[G])$ is ***nontrivially*** *graded with G/G', that is there exist*[a] *such Der_k, $k \in G/G'$, that*

$$Der(\mathbb{C}[G]) = \bigoplus_{k \in G/G'} Der_k \qquad (12)$$

$$[Der_k, Der_l] \subset Der_{kl}$$

[a] arXiv:2308.00512

Main Idea

Introduce a symbol to simplify formulations

Definition

Let d correspond to character χ_d, ∂ correspond to character χ_∂. Denote by $\{\chi_d, \chi_\partial\}$ the character corresponding to $[d, \partial]$.

The following lemma is the essence of the theorem as it shows that the "main"property of gradings hold for Der_k

Lemma

Let $a, b \in G$, d correspond to character $\chi_d : supp\chi_d \subset \Gamma_k$, ∂ correspond to character $\chi_\partial : supp\chi_\partial \subset \Gamma_l$. Then

$$supp\{\chi_d, \chi_\partial\} \subset \Gamma_{kl} \tag{13}$$

It immediatelly follows that $[Der_k, Der_l] \subset Der_{kl}$.
The lemma is purely technical.

Proving the Main Result

It remains to show that $\bigoplus_{k \in G/G'} Der_k$ is indeed direct and that the grading is **non-trivial**. The former follows from decomposition. Recall the example of inner derivations for the latter.

Inner Derivations

For an inner derivation $d_a = [x, a]$ $(a \in G)$ the character is

$$\chi_a(u, v) = \begin{cases} 1, & a = S(u, v), \\ -1, & a = T(u, v), \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Notice that

$$supp \chi_a \subset [a] \subset aG' \tag{15}$$

Since for $k \in G/G'$ is a coset, $k = aG'$ for some $a \in G$. Thus, Der_k contains a non-zero derivation $d_a = [x, a]$. Thus, **all** Der_k are **non-zero**.

Example: Discrete Heisenberg Group

Definition

Discrete Heisenberg Group is

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \tag{16}$$

Consider this group for an example. An interesting event will occur with central derivations.

Statement

All central derivations admit the form ($\alpha, \beta \in \mathbb{C}; z \in \mathbb{Z}$)

$$d_{\alpha,\beta,z}\left(\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right) = (\alpha a + \beta b) \begin{pmatrix} 1 & a & c+z \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

Example: Discrete Heisenberg Group

Facts

It is well-known that

$$Z(\mathbf{H}) = \mathbf{H}' = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Z} \right\} \tag{18}$$

Moreover,

$$\mathbf{H}/\mathbf{H}' = \mathbb{Z} \oplus \mathbb{Z} \tag{19}$$

Therefore, $Der(\mathbb{C}[\mathbf{H}])$ is graded with $\mathbb{Z} \oplus \mathbb{Z}$.

Observation

From (18) it follows that

$$ZDer = Der_{(0,0)} \tag{20}$$

Example: (Non-)Stem Groups and Outer Derivations

Definition

G is a stem group[a] iff

$$Z(G) \leq G' \tag{21}$$

[a] compare with 2-rank nilpotent groups.

$\mathbf{H}$ is stem

$Z(\mathbf{H}) = \mathbf{H}'$

We will need this definition to speak about gradings for outer derivations, that is

Outer Derivations

$OutDer(\mathbb{C}[G]) := Der(\mathbb{C}[G])/InDer(\mathbb{C}[G])$

Example: (Non-)Stem Groups and Outer Derivations

Observation 1

If G is a stem group, then

$$ZDer(\mathbb{C}[G]) \subset Der_0 \tag{22}$$

Observation 2

If G is NOT a stem group, there is an induced (non-trivial) grading of $OutDer$ with G/G'

$$OutDer = \bigoplus_{k \in G/G'} Der_k / InDer_k, \tag{23}$$

where

$$InDer_k := Der_k \cap InDer(\mathbb{C}[G]) \tag{24}$$

That's it, Folks!

Thank you!