Helly-type problems and families of convex sets satisfying (p, q)-property

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Overview

1. Classic results
   - Helly's Theorem
   - Generalizations of Helly's theorem
2. Extension of colorful Helly Theorem
3. Our problem and current progress

Helly's Theorem

Helly's theorem is one of the cornerstones of combinatorial convexity, providing an effective method for determining the intersection pattern of a finite family of convex sets

Theorem 1 (Helly's Theorem)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every d + 1 or fewer sets of them have a non-empty intersection, then $\bigcap \mathcal{F} \neq \varnothing$, where $\bigcap \mathcal{F} := \bigcap\limits_{C \in \mathcal{F}} C$

Generalizations of Helly's theorem

Theorem 2 (fractional Helly)

Let $\alpha \in (0, 1]$ and $d \geq 2$ be fixed. If $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, $|\mathcal{F}| = n$, with at least $\alpha\binom{n}{d+1}$ intersecting d + 1 tuples, then there exists an intersecting subfamily $\mathcal{F}' \subset \mathcal{F}$, with $|\mathcal{F}'| \geq \beta n$, where $\beta > 0$ is a constant that depends only on d and α.

Theorem 3 (colorful Helly)

Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} C_i \neq \varnothing$ for all $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$ then there exists $i \in [d+1]$ such that $\bigcap \mathcal{F}_i \neq \varnothing$

Extension of colorful Helly Theorem

The colorful Helly theorem only provides information that some family has a non-empty intersection, and we do not know anything else about this intersection. Next result provides some additional information on it

Theorem 4 (W.Rao)

Let $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} C_i \neq \varnothing$ for all $C_1 \in \mathcal{F}_1, ..., C_{d+1} \in \mathcal{F}_{d+1}$ then $\sum_{i=1}^{d+1} dim \bigcap \mathcal{F}_i \geq 0$, where $dim\ \varnothing = -1$

Method of Proof of Theorem 4

The proof of Theorem 5 uses reduction to polytopes and considers minimal with respect to inclusion compact convex set U such that families $\mathcal{F}_1 \bigcap U, ..., \mathcal{F}_{d+1} \bigcap U$ satisfy colorful Helly property.

It is easy to see that U is a polytope and every vertex of U is contained in the intersection of some family $\bigcap \mathcal{F}_i$. Then the statement of the theorem is proved by induction on d.

The Problem

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. For $p \geq q \geq d+1$ we say that $\mathcal{F}$ has (p, q) property if among any p sets from $\mathcal{F}$ there are q having a common point

It is well known that for any family $\mathcal{F}$ in $\mathbb{R}^d$ satisfying the (p, q) property there is a set of fixed size $HD_d(p,q)$ that intersects all the sets in $\mathcal{F}$. However, the minimal value of $HD_d(p,q)$ is unknown even for $HD_2(4,3)$. It is only known that $HD_2(4,3) \geq 3$ and $HD_2(4,3) \leq 9$ (Daniel MCGinnis)

The Problem

It is interesting to see if we can apply method of proof of Theorem 4 McGinnis's paper to get a better upper bound on $HD_d(p, q)$ and, as the easiest case, $HD_2(4, 3)$

Current progress

Following the method of proof of Theorem 4, consider family $\mathcal{U}$ of all compact convex sets U such that $\mathcal{F} \bigcap U$ has (4,3)-property.

It is easy to check that $\mathcal{U}$ has minimal element U with respect to inclusion and that U is a polytope.

Then it was proved that for every vertex v of U there are two families C_1, C_2 from $\mathcal{F}$ such that $C_1 \cap C_2 \cap U = \{v\}$. Using this fact it can be shown that if $dimU < 2$ $\mathcal{F}$ can be pierced by 3 points.
In case when $dimU = 2$ we are working on getting an upper bound of $HD_2(4,3)$ using toplogical methods similar to McGinnis's paper

References

[1] Daniel McGinnis "A family of convex sets in the plane satisfying the (4,3)-property can be pierced by nine points"https://arxiv.org/abs/2010.13195