Invariant measures on $\mathbb{R}$ and Fourier transform

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«Hard» problem of measure's theory

Does there exist any measure μ on bounded subsets of $\mathbb{R}^n$ such that:

- $\mu([0,1]^n) = 1$
- If A and B are congruent sets then $\mu(A) = \mu(B)$
- If $E = \bigsqcup_{k=1}^{\infty} E_k$ then $\mu(E) = \sum_{k=1}^{\infty} \mu(E_k)$?

Theorem

Such measure does not exist even in $\mathbb{R}^1$.

«Easy» problem of measure's theory

Does there exist any measure μ on bounded subsets of $\mathbb{R}^n$ such that:

- $\mu([0,1]^n) = 1$
- If A and B are congruent sets, then $\mu(A) = \mu(B)$
- If $E = \bigsqcup_{k=1}^{m} E_k$ then $\mu(E) = \sum_{k=1}^{m} \mu(E_k)$?

Theorem (Banach)

The «Easy» problem of measure's theory has a solution for $\mathbb{R}^1$ and $\mathbb{R}^2$, but it is not unique.

Theorem (Hausdorff)

The «Easy» problem of measure's theory is unsolvable for $\mathbb{R}^n$, $n \geqslant 3$.

Bibliography

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Ultrafilter

Definition

A family $\mathcal{F}$ of subsets of a set is called a filter if

- $\varnothing \notin \mathcal{F}$
- if $B \subset A$ and $B \in \mathcal{F}$ then $A \in F$
- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

An inclusion-wise maximum filter is called the *ultrafilter*.
If $\bigcap_{F \in \mathcal{F}} F = \varnothing$ then the ultrafilter is said to be *non-principal*.

Limit along ultrafilter

Definition

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{R}$.
Let $f \in L_\infty(\mathbb{R}, \mathbb{R})$. We say that $y \in \mathbb{R}$ is a *$\mathcal{F}$-limit of f*, denoted as $y = \lim_{\mathcal{F}} f$, if for every $\varepsilon > 0$ it holds that $\{x : |f(x) - y| \leqslant \varepsilon\} \in \mathcal{F}$.

Lemma

The limit $\lim_{\mathcal{F}} f$ always exists.

Proposition

If $\lim_{x \to \infty} f(x) = a$ then $\lim_{\mathcal{F}} f = a$.

Construction Banach's measure on $\mathbb{R}$

Definition

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{R}$. Let $f \in L_\infty(\mathbb{R})$. The functional $\varphi\colon L_\infty(\mathbb{R}) \to \mathbb{R}$ is defined by

$$\varphi(f) := \lim_{\mathcal{F}} \frac{1}{2x} \int_{-x}^{x} f(t)dt.$$

For $A \in \mathcal{B}(\mathbb{R})$ define $\mu_B(A) := \varphi(I_A)$.

The measure μ_B is shift-invariant, finitely-additive, and finite. The measure μ_B depends on ultrafilter.

Integral

Definition

For a simple function $f(x) = \sum_{i=1}^n c_i I_{A_i}(x)$ define $\int_{\mathbb{R}} f(x) d\mu_B(x) := \sum_{i=1}^n c_i \mu_B(A_i)$.
For $f \in L_\infty(\mathbb{R})$ there are simple functions f_n such that $f_n \to f$ almost everywhere.
Define

$$\int_{\mathbb{R}} f(x) d\mu_B(x) := \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) d\mu_B(x).$$

Lemma

For $f \in L_\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x) d\mu_B(x) = \lim_{\mathcal{F}} \frac{1}{2x} \int_{-x}^{x} f(t) dt.$$

- $L_B^q \subset L_B^p$ for $1 < p < q < \infty$

Inner product and Fourier transform

Definition

Define the inner product $(f, g) := \int_{\mathbb{R}} f(x)\overline{g(x)} d\mu_B(x)$ for $f, g \in L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_B, \mathbb{C})$

- $(e^{i\alpha x}, e^{i\beta x}) = I\{\alpha = \beta\}$. Then $\{e^{i\alpha x}\}$ is the continial orthonormal system
- $(e^{i\alpha x}, e^{ix^2}) = 0$

Definition

Define the Fourier transform $(Ff)(y) := \int_{\mathbb{R}} f(x)e^{-ixy} d\mu_B(x)$.

- $(Fe^{i\alpha x})(y) = I\{y = \alpha\}$
- $(Fe^{i|x|^a})(y) = 0$ for $a > 1$
- $F(f(x + \alpha))(y) = (Ff)(y)e^{i\alpha y}$

Fourier transform

Definition

Define the counting measure $\nu \colon 2^{\mathbb{R}} \to \mathbb{N} \cup \{+\infty\}$ by $\nu(A) := |A|$ for $A \subset \mathbb{R}$.

The measure ν is shift-invariant, locally finite, σ-additive, and not σ-finite.
Consider the Fourier transform F as the functional from $L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_B, \mathbb{C})$ to $L_2(\mathbb{R}, \nu, \mathbb{C})$.
Denote $\mathcal{H}_B^{tr} := \mathrm{Cl}\left\langle e^{i\alpha x}, \alpha \in \mathbb{R}\right\rangle$.

Theorem

$\mathrm{Ker}\, F = (\mathcal{H}_B^{tr})^{\perp}$

Results

The research was carried out on:

- «Hard» and «Easy» problem statement of measure theory and Banach–Tarski paradox
- Notions of Banach's limit, ultrafilter, limit along ultrafilter
- Construction of Banach's measure on $\mathbb{N}$ and $\mathbb{R}$
- Relationship between Banach's and counting measures generated by Fourier transform