Fourier transform on Banach's measure

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Bibliography

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Ultrafilter

Definition

A family $\mathcal{F}$ of subsets of a set is called a filter if

- $\varnothing \notin \mathcal{F}$
- if $B \subset A$ and $B \in \mathcal{F}$ then $A \in F$
- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

An inclusion-wise maximum filter is called the *ultrafilter*.
If $\bigcap_{F \in \mathcal{F}} F \neq \varnothing$ then the ultrafilter is said to be *principal*.

Limit along ultrafilter

Definition

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{R}$.
Let $f \in L_\infty(\mathbb{R}, \mathbb{R})$. We say that $y \in \mathbb{R}$ is a *$\mathcal{F}$-limit of f*, denoted as $y = \lim_{\mathcal{F}} f$, if for every $\varepsilon > 0$ it holds that $\{x : |f(x) - y| \leqslant \varepsilon\} \in \mathcal{F}$.

Lemma

The limit $\lim_{\mathcal{F}} f$ *always exists.*

Proposition

If $\lim_{x\to\infty} f(x) = a$ *then* $\lim_{\mathcal{F}} f = a$.

Construction Banach's measure on $\mathbb{R}$

Definition

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{R}$. Let $f \in L_\infty(\mathbb{R})$. The functional $\varphi\colon L_\infty(\mathbb{R}) \to \mathbb{R}$ is defined by

$$\varphi(f) := \lim_{\mathcal{F}} \frac{1}{2x} \int_{-x}^{x} f(t)dt.$$

For $A \in \mathcal{B}(\mathbb{R})$ define $\mu_B(A) := \varphi(I_A)$.

The measure μ_B is shift-invariant, finitely-additive, and finite.

Integral

Definition

For a simple function $f(x)=\sum_{i=1}^n c_i I_{A_i}(x)$ define $\int_{\mathbb{R}} f(x)d\mu_B(x) := \sum_{i=1}^n c_i \mu_B(A_i)$.
For $f\in L_\infty(\mathbb{R})$ there are simple functions f_n such that $f_n\to f$ almost everywhere.
Define

$$\int_{\mathbb{R}} f(x)d\mu_B(x) := \lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)d\mu_B(x).$$

Lemma

For $f\in L_\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x)d\mu_B(x) = \lim_{\mathcal{F}} \frac{1}{2x}\int_{-x}^{x} f(t)dt.$$

- $L_B^q \subset L_B^p$ for $1<p<q<\infty$.

Inner product and Fourier transform

Definition

Define the inner product $(f,g) := \int_{\mathbb{R}} f(x)\overline{g(x)} d\mu_B(x)$ for $f,g \in L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_B, \mathbb{C})$

- $(e^{i\alpha x}, e^{i\beta x}) = I\{\alpha = \beta\}$. Then $\{e^{i\alpha x}\}$ is the continial orthonormal system.
- $(e^{i\alpha x}, e^{ix^2}) = 0$.

Definition

Define the Fourier transform $(Ff)(y) := \int_{\mathbb{R}} f(x)e^{-ixy} d\mu_B(x)$.

- $(Fe^{i\alpha x})(y) = I\{y = \alpha\}$.
- $(Fe^{i|x|^a})(y) = 0$ for $a > 1$.
- $F(f(x+\alpha))(y) = (Ff)(y)e^{i\alpha y}$.

Fourier transform

Definition

Define the counting measure $\nu \colon 2^{\mathbb{R}} \to \mathbb{N} \cup \{+\infty\}$ by $\nu(A) := |A|$ for $A \subset \mathbb{R}$.

The measure ν is shift-invariant, locally finite, σ-additive, and not σ-finite.
Consider the Fourier transform F as the functional from $L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_B, \mathbb{C})$ to $L_2(\mathbb{R}, \nu, \mathbb{C})$.
Denote $\mathcal{H}_B^{tr} := \mathrm{Cl}\left\langle e^{i\alpha x}, \alpha \in \mathbb{R} \right\rangle$.

Theorem

$\mathrm{Ker}\, F = (\mathcal{H}_B^{tr})^{\perp}$

Plans

- Research averaging of random walks and Fourier transforms of random shifts
- Research counting and Banach's measure on $\mathbb{R}^d$