# Forbidden sunflower conjecture via spread approximation

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In this work we will tell about hypergraphs and sunflowers.

**Definition 1.** *The family $D_1, ..., D_s$ is called a sunflower of size $s$ and with kernel $C$, if $D_i \cap D_j = C$ holds for all $1 \leq i < j \leq s$ (we assume also $D_i \neq D_j$).*

**Definition 2.** $f(n, k, l, s) = max\{|\mathcal{F}| : \mathcal{F} \subset \binom{X}{k}$, *including $\mathcal{F}$ does not contain a sunflower of size $s$ and core size $l$*$\}$

In 1960, it was suggested from Erdos and Rado, that f(n,k,l,s) $< C^k$, for some constant C, depending only on s.

The current good result is the work of 2021 by Alweiss, Lovett, Wu and Zhang with improvements from Tao, namely $f(n, k, l, s) < (Cs \log_2(ks))^k$, where C is some kind of constant (for example, $2^{10}$). It would be great to improve this estimation in some aspect.

Here and below, $\mathcal{F}(S) := \{A \backslash S : A \in \mathcal{F}, S \subset A\}$

**Definition 3.** *For $r \geq 1$, family $\mathcal{F}$ - $r$-spread, if $|\mathcal{F}(X)| \leq r^{-|X|}|\mathcal{F}|, \forall X \subset [n]$*

Also related definition is *homogeneity.*

**Definition 4.** *$\mathcal{F} - \tau$-homogenous, for $\tau > 1$, if $\forall X$ $|\mathcal{F}(X)| \leq \tau^{|X|} \frac{\binom{n-|X|}{k-|X|}}{\binom{n}{k}} |\mathcal{F}|$*

There are some theorems to work with r-spread and homogenous families, for example in [5] or [1].

**Definition 5.** *Given a hypergraph $\mathcal{H} \subset \binom{X}{t}$, we call $\mathcal{H}^+ \subset \binom{X}{k}$ its extension if $\mathcal{H}^+ = \{H \cup A_H : H \in \mathcal{H}\}$ for some pairwise disjoint $A_H \in \binom{X \backslash \operatorname{supp} \mathcal{H}}{k-t}$.*

We assume that $\mathcal{H}$ is a multigraph, i.e. it can contain some edge multiple times. We say that a family $\mathcal{F}$ contains an extension of $\mathcal{H}$ if there exists extension $\mathcal{H}^+$ and permutation $\pi$ of $[n]$ such that $\pi(\mathcal{H}^+) \subset \mathcal{F}$.

Let's assume $n, k, t, s$ satisfy some conditions and given a multihypergraph $\mathcal{H} \subset \binom{[n]}{t}$ of cardinality $s$. In the future, we want to know, if a family $\mathcal{F} \subset \binom{[n]}{k}$ does not contain $\mathcal{H}^+$ (in the context of this work, we are interested in the extension of sunflower), then what is the maximum size it can have? We want to use spread-aproximation technique for this question, prove some lemmas for this, and give a small overview of this task in the current situation.

**Lemma 1.** *Fix positive $n, k, s, t$, such that $n \geqslant k > t$. Let $\mathcal{H} \subset \binom{[n]}{t}$ be a multihypergraph with $s$ edges. If $\mathcal{F} \subset \binom{[n]}{k}$ is $\tau$-homogenous for $\tau < (1 - 1/s)^{-1/t}$,*

$$n > \frac{2^{11}\tau k \log_2(ks)}{\tau^{-t} + s^{-1} - 1} \quad \text{and} \quad n \geqslant \frac{2\tau k t}{\tau^{-t} + s^{-1} - 1},$$

*then $\mathcal{F}$ contains an extension of $\mathcal{H}$.*

**Lemma 2.** *Fix positive integers $n, k, t, q$ such that $n \geqslant k \geqslant t, q > 0$ and real positive $\tau$, $\tau < (1 - s^{-1}/2)^{-1}$. Assume that*

$$n \geqslant \frac{2^{11}\tau k \log_2(ks)}{\tau^{-t} + s^{-1}/2 - 1}, \quad n \geqslant \frac{\tau k(q+t)}{\tau^{-t} - 1 + s^{-1}/2} \quad \text{and} \quad n \geqslant 2sq^2t^2 + q + t.$$

*If there are sets $X_1, \dots, X_s$, pairwise disjoint sets $Y_1, \dots, Y_s$ that do no intersect $\bigcup_i X_i$ and a permutation $\pi$ such that for each $i \in [s]$ we have*

- $X_i \subset \pi(H_i)$,
- $(\pi(H_i) \setminus X_i) \cap \left(\bigcup_j X_j \cup \bigcup_j Y_j\right) = \varnothing$,
- $|X_i \cup Y_i| \leqslant q$,
- *there exists subfamily $\mathcal{F}_i \subset \mathcal{F}$ such that $\mathcal{F}_i(X_i \cup Y_i)$ is $\tau$-homogeneous for $\tau < (1 - s^{-1}/2)^{-1}$,*

*then $\mathcal{F}$ contains an extension of $\mathcal{H}$.*

# References


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