Research on the combination of Top-K and Perm-K gradient sparsification algorithms for distributed setting

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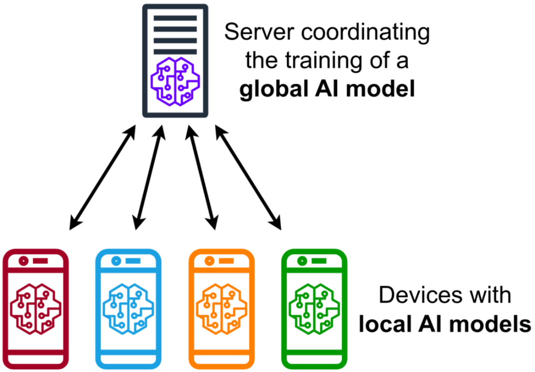
Federated learning



Credit: *Wikipedia*

Figure: Federated Learning scheme

Communication cost is a bottleneck for the Federated Learning approach: worker devices use unstable and slow networks such as WIFI and Cellular.

Brief introduction

- Distributed optimization methods/machine learning methods require the efficient organization of communications, since communications in this case very often take up most of the time of the algorithm.
- To reduce the cost of one communication, you can apply compression of the transmitted information.
- Different Techniques: Random Approaches, Greedy Approaches.
- In this work, the novel method of combining the greedy approach of Top-k and the random approach of Perm-k algorithms for better performance is introduced.

Problem statement

- We consider optimization problems of the form

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\},$$

where $x \in \mathbb{R}^d$ collects the parameters of a statistical model to be trained, n is the number of workers/devices, and $f_i(x)$ is the loss incurred by model x on data stored on worker i.

- A general baseline for solving problem is distributed gradient descent, performing updates of the form

$$x^{k+1} = x^k - \frac{\eta^k}{n} \sum_{i=1}^{n} \nabla f_i \left(x^k \right),$$

where $\eta^k > 0$ is a stepsize.

Compressors review

- **Paper:** On Biased Compression for Distributed Learning (Aleksandr Beznosikov, Samuel Horváth, Peter Richtárik, Mher Safaryan)
- **Main contribution:** Distributed SGD with Biased Compression and Error Feedback Algorithm

Definition

Top-k

$$\mathcal{C}(x) := \sum_{i=d-k+1}^{d} x_{(i)} e_{(i)}$$

where coordinates are ordered by their magnitudes so that $|x_{(1)}| \leq |x_{(2)}| \leq \cdots \leq |x_{(d)}|$.

Definition

Error Feedback $e_i^{k+1} = e_i^k + \nabla f_i(x^k) - C(e_i^k + \nabla f_i(x^k))$

Compressors review

- **Paper:** Permutation Compressors for Provably Faster Distributed Nonconvex Optimization (Rafał Szlendak, Alexander Tyurin, Peter Richtárik)
- **Main contribution:** Construction of the new compressors based on the idea of a random permutation (Perm K).
  Provably reduce the variance caused by compression beyond what independent compressors can achieve.

Definition

(Perm K for $d \geq n$). Assume that $d \geq n$ and $d = qn$, where $q \geq 1$ is an integer. Let $\pi = (\pi_1, \dots, \pi_d)$ be a random permutation of $\{1, \dots, d\}$. Then for all $x \in \mathbb{R}^d$ and each $i \in \{1, 2, \dots, n\}$ we define

$$\mathcal{C}_i(x) := n \cdot \sum_{j=q(i-1)+1}^{qi} x_{\pi_j} e_{\pi_j}.$$

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Biased classes

Biased compressor class

We say $C \in \mathbb{B}^3(\delta)$ for some $\delta > 1$ if

$$\mathrm{E}\left[\|C(x) - x\|_2^2\right] \leq \left(1 - \frac{1}{\delta}\right)\|x\|_2^2, \quad \forall x \in \mathbb{R}^d$$

Bounds for compressors

- *TopK:* $(1 - \frac{1}{\delta}) = \frac{d-k}{d}$ [Alistarh et al., 2018a]

Error feedback proof

Lemma 22 [Beznosikov et al. 2020]

$\eta^k \leq \frac{1}{14(2\delta+B)L}, \forall k \geq 0$ and $\left\{\left(\eta^k\right)^2\right\}_{k\geq 0}$ – 2δ-slow decreasing. Then

$$\mathrm{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} e_i^{k+1}\right\|_2^2\right] \leq \frac{(1-1/\delta)}{49L(2\delta+B)}\sum_{j=0}^{k}\left[\left(1-\frac{1}{4\delta}\right)^{k-j}\left(f\left(x^j\right)-f\left(x^*\right)\right)\right]+$$

$$+\eta^k \frac{2(\delta-1)}{7L}\left(2D+\frac{C}{2\delta+B}\right).$$

Furthermore, for any 4δ-slow increasing non-negative sequence $\left\{w^k\right\}_{k\geq 0}$ it holds:

$$3L\cdot\sum_{k=0}^{K} w^k \cdot \mathrm{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} e_i^k\right\|_2^2\right] \leq$$

$$\leq \frac{1}{4}\sum_{k=0}^{K} w^k\left(\mathrm{E}\left[f\left(x^k\right)\right]-f\left(x_*\right)\right)+\left(3\delta D+\frac{3C}{4}\right)\sum_{k=0}^{K} w^k\eta^k.$$

Proof [Beznosikov et al. 2020]

$$\mathrm{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} e_i^{k+1}\right\|_2^2\right] \overset{Jensen}{\leq} \frac{1}{n}\mathrm{E}\left[\sum_{i=1}^{n}\left\|e_i^{k+1}\right\|_2^2\right] =$$

$$= \frac{1}{n}\mathrm{E}\left[\sum_{i=1}^{n}\left\|e_i^k + \eta^k g_i^k - \tilde{g}_i^k\right\|_2^2\right] =$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left[\left\|e_i^k + \eta^k g_i^k - \mathcal{C}\left(e_i^k + \eta^k g_i^k\right)\right\|_2^2\right] \overset{\mathbb{B}(\delta)}{\leq}$$

$$\overset{\mathbb{B}(\delta)}{\leq} \frac{1 - 1/\delta}{n}\sum_{i=1}^{n}\mathrm{E}_{\nabla}\left[\left\|e_i^k + \eta^k g_i^k\right\|_2^2\right] =$$

$$= \frac{1 - 1/\delta}{n}\sum_{i=1}^{n}\mathrm{E}_{\nabla}\left[\left\|e_i^k + \eta^k \nabla f_i\left(x^k\right) + \eta^k \xi_i^k\right\|_2^2\right]$$

Proof [NEW]

$$\mathrm{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} e_i^{k+1}\right\|_2^2\right] \overset{definition}{=}$$

$$= \mathrm{E}\left[\left\|\frac{1}{n}\sum_{i=1}^{n} e_i^{k} + \eta^k g_i^k - \mathcal{C}\left(e_i^k + \eta^k g_i^k\right)\right\|_2^2\right] =$$

$$= \mathrm{E}\left[\left\|\frac{1}{n}\left(\sum_{i=1}^{n} \mathcal{C}\left(e_i^k + \eta^k g_i^k\right)\right) - \frac{1}{n}\left(\sum_{i=1}^{n} e_i^k + \eta^k g_i^k\right)\right\|_2^2\right] \overset{y_i := e_i^k + \eta^k g_i^k}{=}$$

$$= \mathrm{E}\left[\left\|\frac{1}{n}\left(\sum_{i=1}^{n} \mathcal{C}\left(y_i\right)\right) - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)\right\|_2^2\right] \overset{?}{\leq} (1-\delta)\left\|\sum_{i=1}^{n} y_i\right\|^2$$

Optimization problem

Main lemma

$$\mathrm{E}\left[\left\|\frac{1}{n}\left(\sum_{i=1}^{n} TopK\left(Perm_i\left(y_i\right)\right)\right)-\frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)\right\|_2^2\right] \overset{?}{\leq} (1-\delta)\left\|\sum_{i=1}^{n} y_i\right\|^2$$

Main lemma (simple case)

Lets assume, that $\forall i, j \nabla f_i(x) = \nabla f_j(x) = y$.
Then the following implies:

$$\mathbb{E}\|\frac{1}{n}\sum_{i}^{n} Top_k(Perm_i(y)) - y\|^2 \leq (1-\frac{nk}{d})\|y\|^2$$

This is n times better than better than $(1-\frac{k}{d})$. [Beznosikov et al. 2020]

Proof. 1. Lets proof the inequality itself. Fix the norm $\|y\|^2 = const$. Without loss of generality, y coordinates are in increasing order $y_1 < y_2 < \ldots < y_d$.

$$\mathbb{E}\|\frac{1}{n}\sum_i^n Top_k(Perm_i(y)) - y\|^2 = \frac{1}{\#\sigma}\sum_\sigma \frac{1}{n^2}\|\sum_i^n Top_k(Perm_i(y)) - ny\|^2 = \frac{1}{\#\sigma}\frac{1}{n^2}\sum_\sigma\sum_j^d (nI^\sigma_{y_j}y_j - ny_j)^2$$

Where $I^\sigma_{y_j} = 1$ if y_j is chosen by at least one Top_k in the σ permutation of $Perm$-s, and 0 otherwise. Then this for each j in fixed permutation σ we have:

$$(nI^\sigma_{y_j} - n)^2 = \begin{cases} 0 & \text{if } y_j \text{ is not chosen} \\ n^2, & \text{otherwise} \end{cases}$$

For each $1 \leq j \leq d$ let $p_j = \sum_\sigma (nI^\sigma_{y_j} - ny_j)^2$. Then it is clear that $\forall i < j \; p_i \geq p_j$, because greater the value of y_j more often it is chosen by Top_k.

$$\sum_\sigma\sum_j^d (nI^\sigma_{y_j}y_j - ny_j)^2 = \sum_j^d\sum_\sigma (nI^\sigma_{y_j}y_j - ny_j)^2 = \sum_j^d p_j y_j^2$$

Lets look at $y_i < y_{i+1}$. If we increase y_i^2 by ε and decrease y_{i+1}^2 by the same value, then $(y_i^2+\varepsilon)+(y_{i+1}^2-\varepsilon) = y_i^2 + y_{i+1}^2$. But $p_i(y_i^2+\varepsilon) + p_{i+1}(y_{i+1}^2-\varepsilon) \geq p_i y_i^2 + p_{i+1}y_{i+1}^2$. If $0 < y_i < y_{i+1}$ or $y_i < y_{i+1} < 0$, then moving y_i and y_{i+1} towards each other only increases variance value. So we can move all coordinates, until we have $y_1 = y_2 = \ldots = y_m < 0 < y_{m+1} = \ldots = y_d$.
Without loss of generality, $\|y_1\|^2 \leq \|y_{m+1}\|^2$.

2. It stands that the case, when $y_1 = y_2 = \ldots = y_d$ is optimal for maximizing the variance with $\|y\|^2 = const$. In that case, it can be easily observed, that the constant is $1 - \frac{nk}{d}$.

$$\mathbb{E}\|\frac{1}{n}\sum_i^n Top_k(Perm_i(y)) - y\|^2 = \sum_j^{\text{not chosen } d-nk \text{ coordinates}} y_j^2 = (d-nk)y_1^2 = (1 - \frac{nk}{d})\|y\|^2$$

□

Hessian variance

Hessian variance

Let $L_\pm \geq 0$ be the smallest quantity such that

$$\frac{1}{n}\sum_{i=1}^{n}\|\nabla f_i(x) - \nabla f_i(y)\|^2 - \|\nabla f(x) - \nabla f(y)\|^2 \leq L_\pm^2 \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^d.$$

We can refer to the quantity $L_\pm^2$ by the name Hessian variance.

- Theoretical proof will help us to find the best μ parameter depending on other hyperparameters, including $L_\pm$.
- The following attempts were to estimate the communication complexity of the algorithm on the problem where $L_\pm = 0$.

General case

Main lemma

$$\mathrm{E}\left[\left\|\frac{1}{n}\left(\sum_{i=1}^{n} TopK\left(Perm_i\left(y_i\right)\right)\right) - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)\right\|_2^2\right] \overset{?}{\leq} (1-\delta)\left\|\sum_{i=1}^{n} y_i\right\|^2$$

Gradient's coefficients

In case when $\nabla f(x) = y$, assuming that $y_1 < y_2 < \ldots < y_d$. We want to count the probability P_i - that y_i is chosen by $TopK(Perm(y))$ in all possible permutations.

$$P_i = \frac{\sum_{j=0}^{k-1} \mathbb{C}_{i-1}^{j} \cdot \mathbb{C}_{d-i}^{\frac{d}{n}-1-j}}{\mathbb{C}_{d-1}^{\frac{d}{n}-1}}$$

Further considerations

- We can try solving this quadratic optimization problem by using numerical solvers. This will help us to find the dependence on $L_{\pm}$, n, k, d.
  Bottleneck: exponential complexity with increase of n and d!
- We may test the hypothesis, that inequation's optimum is the case of equal gradients. For that $TopK(Perm_i) = Perm_i$ case can be checked. Also numerical results will give insights.
- Easier cases can be checked.
  For example, when $\forall i, j \, \|\nabla f_i - \nabla f_j\|^2 \leq \mathcal{C}$ for $\mathcal{C} \in \mathbb{R}^+$

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Observable compressors and EFs

TopK + Error Feedback

$$\mathcal{C}(x) := \mathcal{T}_k(x)$$

Unbiased TopK-PermK

$$\mathcal{C}(x) := \mathcal{T}_k \circ \mathcal{P}_q(x) \cdot n$$

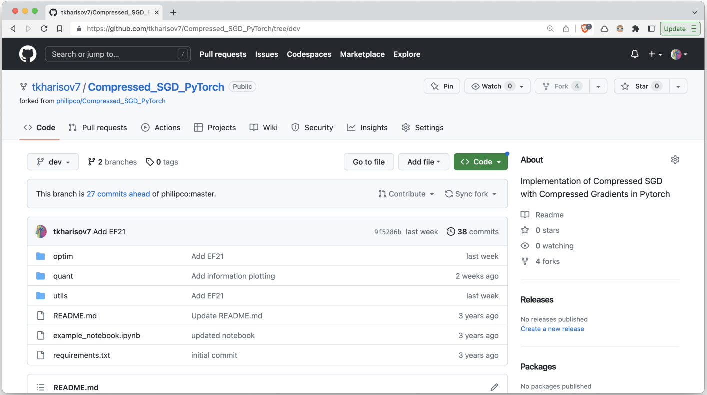
Biased TopK-PermK + Error Feedback

$$\mathcal{C}(x) := \mathcal{T}_k \circ \mathcal{P}_q(x)$$

Classic Error Feedback

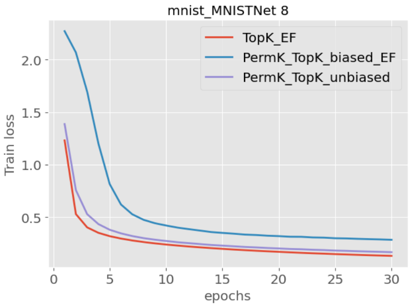
$$e_i^{k+1} = e_i^k + \nabla f_i(x^k) - C(Q_i^k(e_i^k + \nabla f_i(x^k)))$$

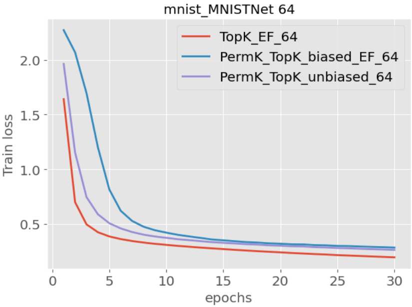
Framework implementation



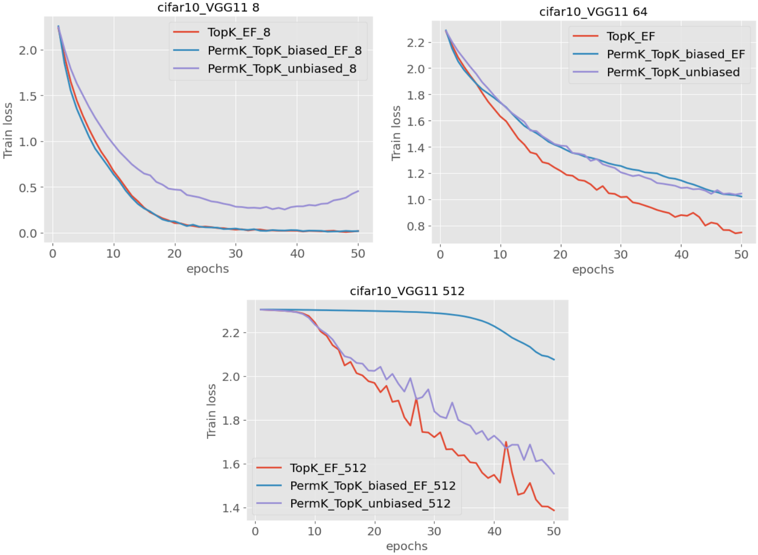
We have developed framework for our optimization problems and algorithm. Implementation is based on Horthath et al. 2020.

MnistNet comparison

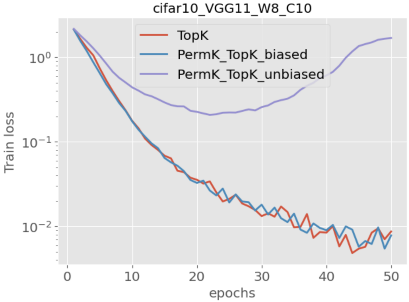


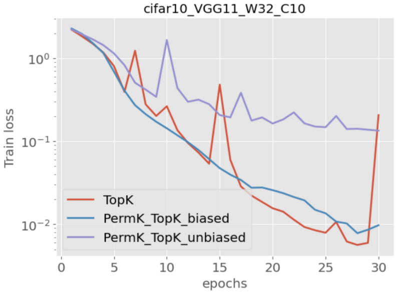


CIFAR-10 + VGG11



Dataset mix-up (common 10%)





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Results

1. It was rigorously proven, that in $L_{\pm} = 0$ regime TopK+PermK combined with the classic Error Feedback achieves n times better convergence results than TopK with Error Feedback.
2. A theoretical search of convergence rate in a general regime is equivalent to the quadratic optimization task with $d * n$ variables and $d! * n$ inequalities.
3. Experiments prove that with an increase in datasets correlation, TopK-PermK perfomance enjoys a similar convergence rate as TopK-EF.
4. With the increase of nodes number TopK-PermK without Error Feedback shows better perfomance than one with Error Feedback.

References

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Q&A

Your questions, please!