

# Sign operator for $(L_0, L_1)$ –smooth optimization

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This presentation explores  $(L_0, L_1)$ -smooth optimization, a generalization of traditional smoothness for functions with sparse or structured gradients. Topics include:

- Definition of  $(L_0, L_1)$ -smoothness and its implications.
- Key algorithms, such as Sign-SGD and its variants.
- Theoretical results under heavy-tailed (HT) noise.

## Abstract

In Machine Learning, the non-smoothness of optimization problems, the high cost of communicating gradients between workers, and severely corrupted data during training necessitate generalized optimization approaches. This paper explores the efficacy of sign-based methods, which address slow transmission by communicating only the sign of each minibatch stochastic gradient. We investigate these methods within  $(L_0, L_1)$ -smooth problems, which encompass a wider range of problems than the  $L$ -smoothness assumption. Furthermore, under the assumptions above, we investigate techniques to handle heavy-tailed noise, defined as noise with bounded  $\kappa$ -th moment  $\kappa \in (1, 2]$ . This includes the use of SignSGD with Majority Voting in the case of symmetric noise. We then attempt to extend the findings to convex cases using error feedback.

## Definition of $(L_0, L_1)$ -Smoothness

A function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $(L_0, L_1)$ -smooth if, for all  $x, y \in \mathbb{R}^d$ , its gradient satisfies:

$$\|\nabla f(x) - \nabla f(y)\| \leq \left( L_0 + L_1 \sup_{u \in [x, y]} \|\nabla f(u)\| \right) \|x - y\|$$

where:

- $L_0 \geq 0$ : Base Lipschitz constant.
- $L_1 \geq 0$ : Gradient-dependent smoothness factor.
- $[x, y]$ : Line segment between  $x$  and  $y$ .

This captures non-uniform gradient behavior in sparse or noisy optimization.

## Examples of $(L_0, L_1)$ -Smooth Functions

The following functions illustrate  $(L_0, L_1)$ -smoothness:

- Let  $f(x) = \|x\|^{2n}$ , where  $n$  is a positive integer. Then,  $f(x)$  is convex and  $(2n, 2n - 1)$ -smooth. Moreover,  $f(x)$  is not  $L$ -smooth for  $n \geq 2$  and any  $L \geq 0$ .
- $f(x) = \log(1 + \exp(-a^\top x))$ , where  $a \in \mathbb{R}^d$  is some vector. It is known that this function is  $L$ -smooth and convex with  $L = \|a\|^2$ . However, one can show that  $f$  is also  $(L_0, L_1)$ -smooth with  $L_0 = 0$  and  $L_1 = \|a\|$ . For  $\|a\| \gg 1$ , both  $L_0$  and  $L_1$  are much smaller than  $L$ .

These are relevant to compressed sensing and machine learning.

## Algorithm 1 SignSGD

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**Input:** Starting point  $x^1 \in \mathbb{R}^d$ , number of iterations  $T$ , stepsizes  $\{\gamma_k\}_{k=1}^T$ .

1: **for**  $k = 1, \dots, T$  **do**

2:    Sample  $\xi^k$  and compute estimate  $g^k = \nabla f(x^k, \xi^k)$ ;

3:    Set  $x^{k+1} = x^k - \gamma_k \cdot \text{sign}(g^k)$ ;

4: **end for**

**Output:** uniformly random point from  $\{x^1, \dots, x^T\}$ .

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### Algorithm 2 minibatch-SignSGD

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**Input:** Starting point  $x^1 \in \mathbb{R}^d$ , number of iterations  $T$ , stepsizes  $\{\gamma_k\}_{k=1}^T$ , batchsizes  $\{B_k\}_{k=1}^T$ .

- 1: **for**  $k = 1, \dots, T$  **do**
- 2:     Sample  $\{\xi_i^k\}_{i=1}^{B_k}$  and compute gradient estimate  
$$g^k = \sum_{i=1}^{B_k} \nabla f(x^k, \xi_i^k) / B_k;$$
- 3:     Set  $x^{k+1} = x^k - \gamma_k \cdot \text{sign}(g^k)$ ;
- 4: **end for**

**Output:** uniformly random point from  $\{x^1, \dots, x^T\}$ .

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## Algorithm 4 M-SignSGD

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**Input:** Starting point  $x^1 \in \mathbb{R}^d$ , number of iterations  $K$ ,  
stepsizes  $\{\gamma_k\}_{k=1}^T$ , momentums  $\{\beta_k\}_{k=1}^T$ .

1: **for**  $k = 1, \dots, T$  **do**

2:    Sample  $\xi^k$  and compute estimate  $g^k = \nabla f(x^k, \xi^k)$ ;

3:    Compute  $m^k = \beta_k m^{k-1} + (1 - \beta_k)g^k$ ;

4:    Set  $x^{k+1} = x^k - \gamma_k \cdot \text{sign}(m^k)$ ;

5: **end for**

**Output:** uniformly random point from  $\{x^1, \dots, x^T\}$ .

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**Algorithm 3** MajorityVote-SignSGD

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**Input:** Starting point  $x^0 \in \mathbb{R}^d$ , number of iterations  $T$ ,  
stepsizes  $\{\gamma_k\}_{k=1}^T$ , batchsizes  $\{M_k\}_{k=1}^T$ .

- 1: **for**  $k = 1, \dots, T$  **do**
- 2:   Sample  $\{\xi_i^k\}_{i=1}^{B_k}$  and compute gradient estimate  
$$g^k = \sum_{i=1}^{M_k} \text{sign}(\nabla f(x^k, \xi_i^k));$$
- 3:   Set  $x^{k+1} = x^k - \gamma_k \cdot \text{sign}(g^k);$
- 4: **end for**

**Output:** uniformly random point from  $\{x^1, \dots, x^T\}$ .

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## HT Noise Definition

The unbiased estimate  $\nabla f(x, \xi)$  has bounded  $\kappa$ -th moment  $\kappa \in (1, 2]$  for each coordinate, i.e.,  $\forall x \in \mathbb{R}^d$ :

- $E_\xi[\nabla f(x, \xi)] = \nabla f(x)$ ,
- $E_\xi[|\nabla f(x, \xi)_i - \nabla f(x)_i|^\kappa] \leq \sigma_i^\kappa, i \in \overline{1, d}$ ,

where  $\vec{\sigma} = [\sigma_1, \dots, \sigma_d]$  are non-negative constants. If  $\kappa = 2$ , then the noise is called a bounded variance.

# Assumptions

## Assumption (Lower Bound)

*The function  $f$  is lower bounded:  $f(x) \geq f^* > -\infty, \forall x \in \mathbb{R}^d$ .*

## Assumption (Smoothness)

*The function  $f$  is differentiable and  $(L_0, L_1)$ -smooth:*

$$\|\nabla f(x) - \nabla f(y)\| \leq \left( L_0 + L_1 \sup_{u \in [x,y]} \|\nabla f(u)\| \right) \|x - y\|$$

## Assumption (Heavy-Tailed Noise)

*The gradient estimate  $\nabla f(x, \xi)$  is unbiased with bounded  $\kappa$ -th moments:*

- $\mathbb{E}_\xi[\nabla f(x, \xi)] = \nabla f(x)$ ,
- $\mathbb{E}_\xi[|\nabla f(x, \xi)_i - \nabla f(x)_i|^\kappa] \leq \sigma_i^\kappa, i = 1, \dots, d$ ,

*where  $\kappa \in (1, 2]$ ,  $\sigma_i \geq 0$ .*

# Theorem for Sign-SGD-minibatch

**Theorem 1 (HP complexity for minibatch-SignSGD).**

Consider lower-bounded  $L$ -smooth function  $f$  (As. 1, 2) and HT gradient estimates (As. 3). Then Alg. 2 requires the sample complexity  $N$  to achieve  $\frac{1}{T} \sum_{k=1}^T \|\nabla f(x^k)\|_1 \leq \varepsilon$  with probability at least  $1 - \delta$  for:

**arbitrary tuning:**  $T, \gamma_k \equiv \frac{\gamma_0}{\sqrt{T}}, B_k \equiv \max\{1, B_0 T\}$ :

$$N = O \left( \frac{B_0 (\Delta_1 / \gamma_0 + L d \gamma_0)^4}{\varepsilon^4} + \frac{1}{B_0} \left( \frac{\|\vec{\sigma}\|_1}{\varepsilon} \right)^{\frac{2\kappa}{\kappa-1}} \right), \quad (3)$$

**optimal tuning:**  $T = O \left( \frac{\Delta_1 L_\delta d}{\varepsilon^2} \right), \gamma_k \equiv \sqrt{\frac{\Delta_1}{8 L_\delta d T}}, B_k \equiv \max \left\{ 1, \left( \frac{16 \|\vec{\sigma}\|_1}{\varepsilon} \right)^{\frac{\kappa}{\kappa-1}} \right\} :$

$$N = O \left( \frac{\Delta_1 L_\delta d}{\varepsilon^2} + \frac{\Delta_1 L_\delta d}{\varepsilon^2} \left( \frac{\|\vec{\sigma}\|_1}{\varepsilon} \right)^{\frac{\kappa}{\kappa-1}} \right), \quad (4)$$

where  $\Delta_1 = f(x^1) - f^*$ ,  $L_\delta = L \log(1/\delta)$ .

# Theorem for Sign-SGD-majority-Vote

**Theorem 2 (HP complexity for MajorityVote-SignSGD).**

Consider lower-bounded  $L$ -smooth function  $f$  (As. 1, 2) and the gradient estimates corrupted by **unimodal and symmetric** HT noise (As. 3). Then Alg. 3 requires the sample complexity  $N$  to achieve  $\frac{1}{T} \sum_{k=1}^T \|\nabla f(x^k)\|_1 \leq \varepsilon$  with probability at least  $1 - \delta$  for:

**arbitrary tuning:**  $T, \gamma_k \equiv \frac{\gamma_0}{\sqrt{T}}, M_k \equiv \max\{1, M_0 T\}$ :

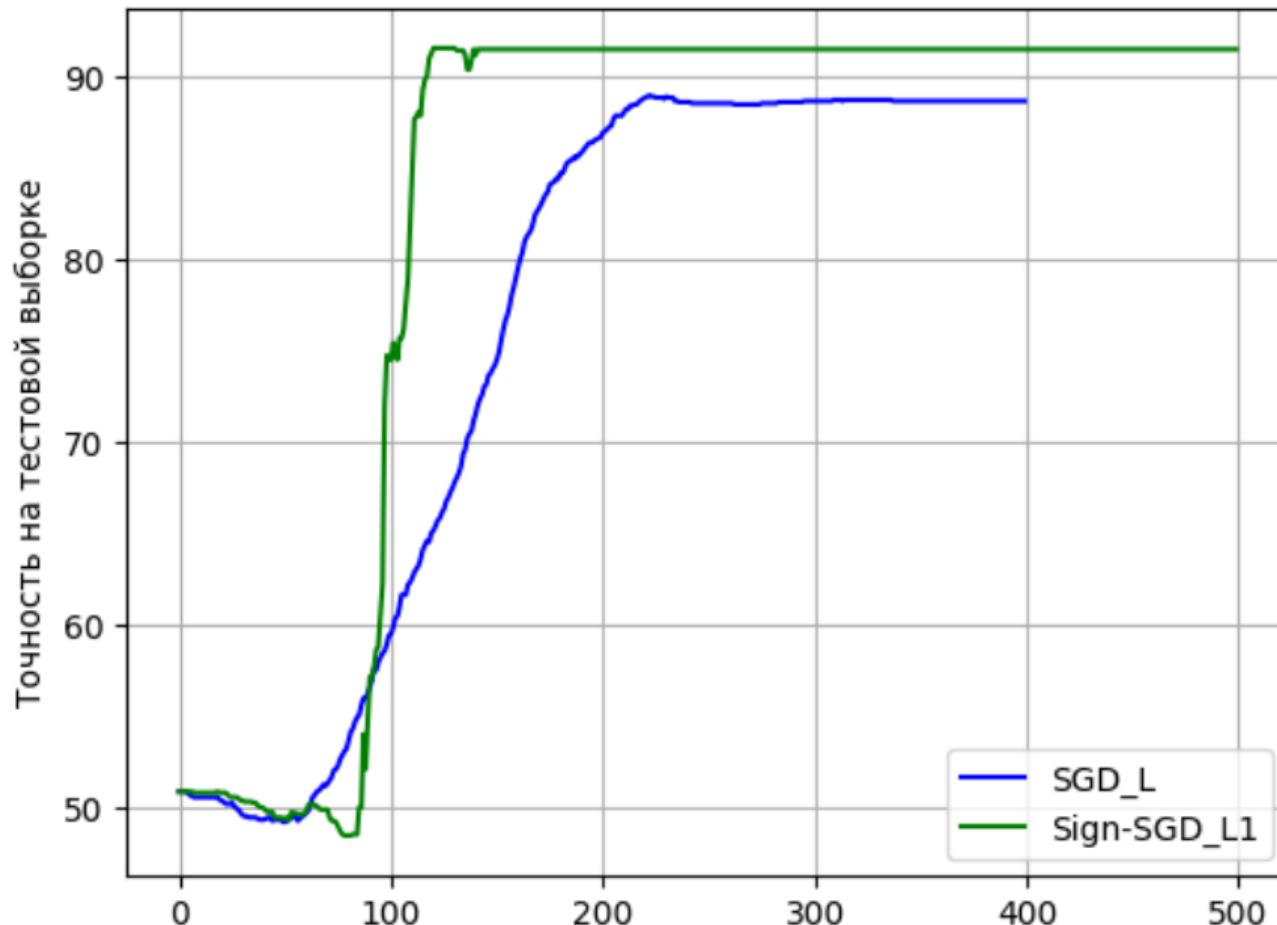
$$N = O \left( \frac{M_0 (\Delta_1 / \gamma_0 + L_\delta d \gamma_0 + a_\kappa \|\vec{\sigma}\|_1 / \sqrt{M_0})^4}{\varepsilon^4} \right), \quad (7)$$

**optimal tuning:**  $T = O \left( \frac{\Delta_1 L_\delta d}{\varepsilon^2} \right), \gamma_k \equiv \sqrt{\frac{\Delta_1}{8 L_\delta d T}}, M_k \equiv \max \left\{ 1, \left( 8 a_\kappa \frac{\|\vec{\sigma}\|_1}{\varepsilon} \right)^2 \right\} :$

$$N = O \left( \frac{\Delta_1 L_\delta d}{\varepsilon^2} + \frac{\Delta_1 L_\delta d}{\varepsilon^2} \left( \frac{a_\kappa \|\vec{\sigma}\|_1}{\varepsilon} \right)^2 \right), \quad (8)$$

where  $\Delta_1 = f(x^1) - f^*$ ,  $L_\delta = L \log^{1/\delta} (a_\kappa)^\kappa := \frac{\kappa+1}{\kappa-1}$ .

## Изменение точности во время обучения



# Key Articles for Research

Topic	Title	Year	Authors	Paper	Summary
Key article 1	Sign Operator for Coping with Heavy-Tailed Noise	2025	Kornilov et al.	arXiv	Proofs for heavy-tailed noise
Key article 2	signSGD: Compressed Optimisation for Non-Convex Problems	2018	J. Bernstein et al.	PMLR	3 Sign-based methods
Key article 3	Methods for Convex (L0,L1)-Smooth Optimization: Clipping, Acceleration, and	2024	Gorbunov et al.	arXiv	New convergence guarantees for existing methods

# Additional Papers for Methods and Proofs (Part 1)

Topic	Title	Year	Authors	Paper	Summary
Additional theory	Robustness to Unbounded Smoothness of Generalized SignSGD	2022	M. Crawshaw et al.	Curran Associates	$L_0, L_1$ SignSGD
Additional theory	Error Feedback Fixes SignSGD and other	2019	Karimireddy et al.	PMLR	Check for convex case

## Additional Papers for Methods and Proofs (Part 2)

Topic	Title	Year	Authors	Paper	Summary
Proofs	From Gradient Clipping to Normalization for Heavy Tailed SGD	2024	Hubler et al.	arXiv	Heavy Tailed SGD
Additional theory	Why gradient clipping accelerates training: A theoretical justification for adaptivity	2020	Zhang et al.	arXiv	Intro to (L0,L1)-smoothness as- sump.