Measure invariance under a group of transforms

E. Dzhenzher

«Hard» problem of measure's theory

Does there exist any measure μ on bounded subsets of $\mathbb{R}^n$ such that:

- $\mu([0,1]^n) = 1$
- If A and B are congruent sets then $\mu(A) = \mu(B)$
- If $E = \bigsqcup_{k=1}^{\infty} E_k$ then $\mu(E) = \sum_{k=1}^{\infty} \mu(E_k)$?

Theorem

Such measure does not exist even in $\mathbb{R}^1$.

«Easy» problem of measure's theory

Does there exist any measure μ on bounded subsets of $\mathbb{R}^n$ such that:

- $\mu([0,1]^n) = 1$
- If A and B are congruate sets, then $\mu(A) = \mu(B)$
- If $E = \sqcup_{k=1}^{m} E_k$ then $\mu(E) = \sum_{k=1}^{m} \mu(E_k)$?

Theorem (Banach)

The «Easy» problem of measure's theory has a solution for $\mathbb{R}^1$ and $\mathbb{R}^2$, but it is not unique.

Theorem (Hausdorff)

The «Easy» problem of measure's theory is unsolvable for $\mathbb{R}^n$, $n \geqslant 3$.

Banach's limit

Definition

A functional $B\colon \ell_\infty \to \mathbb{R}$ is called *the Bahach's limit* if

- $B(1) = 1$
- $B(x) \geqslant 0$ for any $x \geqslant 0$
- $B(Tx) = B(x)$ for any x where $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$.

Ultrafilter

Definition

A family $\mathcal{F}$ of subsets of a set is called a filter if

- $\varnothing \notin \mathcal{F}$
- if $B \subset A$ and $B \in \mathcal{F}$ then $A \in F$
- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

An inclusion-wise maximum filter is called the *ultrafilter*.
If $\bigcap_{F \in \mathcal{F}} F \neq \varnothing$ then ultrafilter is said to be *principal*.

Limit along ultrafilter

Definition

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$.
Let (X, d) be a metric space and $x \in X$. Let $\{x_n\} : \mathbb{N} \to X$ be a bounded sequence of points. We say that x is a *$\mathcal{F}$-limit of x_n*, denoted as $x = \lim_{\mathcal{F}} x_n$, if for every $\varepsilon > 0$ it holds that $\{n : d(x_n, x) \leqslant \varepsilon\} \in \mathcal{F}$.

Construction of Banach limit on ℓ_∞ and measure on $\mathbb{N}$

Theorem

Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$. Let $x = (x_n)$ be a bounded real sequence. Then the functional $\varphi\colon \ell_\infty \to \mathbb{R}$ defined by

$$\varphi(x) := \lim_{\mathcal{F}} \frac{x_1 + x_2 + \ldots + x_n}{n}$$

is a Banach limit.

Define $\ell\colon 2^{\mathbb{N}} \to \ell_\infty$ by $\ell(A) := x$ where $x_n = 1$ if $n \in A$ and $x_n = 0$ otherwise.
Then define $\mu_B(A) := \varphi(\ell(A))$.
The measure μ_B is shift-invariant, locally finite, finitely-additive, σ-finite.

Plans

- Learn base of Banach limits in ℓ_∞ and $L_\infty(\mathbb{R})$
- Construct spaces L_p with Banach measure and obtain properties of Fourier transform in these spaces.