Targeted College Admission

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The Stable Marriage Problem

The Stable Marriage problem was considered and solved by Gale and Shapley. Given two sets M and W (of men and women), where $|M| = |W| = n$, a pair $(m, w) \in M \times W$ is called a *family*. For each individual, there is also given a complete ranking of the individuals in the other group. More precisely, we write $w_j \succ_{m_i} w_k$ for $m_i \in M$, $w_j, w_k \in W$, if man m_i prefers woman w_j to w_k, and similarly, $m_j \succ_{w_i} m_k$, $w_i \in W$, $m_j, m_k \in M$, if woman w_i prefers m_j to m_k. It is assumed that for each individual, the corresponding preference forms a complete ordering of the individuals in the other group.

A *matching* is a set of n families such that each individual in $M \cup W$ belongs to exactly one of these families. Given a matching $\mathcal{M}$, a family F is said to be *breaking* (with respect to $\mathcal{M}$) if both members of F prefer each other more than their respective partners in the families they belong to in $\mathcal{M}$. The matching $\mathcal{M}$ is called *stable* if there exists no breaking family with respect to it.

The Stable Marriage Problem

The *Stable Matching Theorem* of Gale and Shapley claims that such a stable matching always exists and can be found in $O(n^2)$ time.

The above result of Gale and Shapley has inspired a significant amount of research, relating to algorithms, the lattice structure of solutions, strategic behavior, and more. The wide range of applications spans from student admissions and the assignment of medical interns and residents to economics, social sciences, and discrete mathematics.

In our research, we are interested in generalizing the Stable Marriage Theorem for "families"consisting of three individuals.

Nonexistence of stable threesome matchings

Unfortunately, if we have no restrictions on the preferences of the agents, a stable matching does not exist. This was proven by Ahmet Alkan in 1987

A:

	a	b	c
$\tilde{A}$	3		1
$\tilde{B}$			2
$\tilde{C}$			

B:

	a	b	c
$\tilde{A}$			
$\tilde{B}$		2	1
$\tilde{C}$			3

C:

	a	b	c
$\tilde{A}$			
$\tilde{B}$			
$\tilde{C}$			1

$\tilde{A}$:

	a	b	c
A	1		
B			
C			

$\tilde{B}$:

	a	b	c
A			2
B		3	1
C			

$\tilde{C}$:

	a	b	c
A			
B			1
C			2

a:

	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
A	1		
B			
C			

b:

	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
A			
B		1	
C			

c:

	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$
A		3	1
B			2
C			4

Fig. 1. Preferences of agents.

Existence of stable matchings in some three-sided systems

A natural step towards solving this problem is to weaken the preferences. Danilov proved that stable matchings always exist for the special case of certain **acyclic preferences**.

Suppose that preferences are such that every man is interested in the first place in women, and every woman is interested in the first place in men. Then there exist stable three-sided matchings.

Stable matchings in three-sided systems with cyclic preferences

Danilov also raised the question of the existence of a stable matching in the case of **lexicographically cyclic preferences**, where men primarily care about women, women primarily care about cats, and cats primarily care about men. In 2004, Boros demonstrated that the answer is still negative by constructing a three-sided system with lexicographically cyclic preferences for which no stable matching exists.

$$
\begin{aligned}
&(w_1, c_i) \succ_m (w_2, c_j) \succ_m (w_3, c_k) && \text{for } m \in \{m_1, m_3\}, \\
&(w_2, c_i) \succ_m (w_3, c_j) \succ_m (w_1, c_k) && \text{for } m = m_2, \\
&(w_i, c_2) \succ_m (w_i, c_1) \succ_m (w_i, c_3) && \text{for } m \in \{m_1, m_2, m_3\}; \\
\\
&(c_1, m_i) \succ_w (c_2, m_j) \succ_w (c_3, m_k) && \text{for } w \in \{w_1, w_3\}, \\
&(c_2, m_i) \succ_w (c_3, m_j) \succ_w (c_1, m_k) && \text{for } w = w_2, \\
&(c_i, m_2) \succ_w (c_i, m_1) \succ_w (c_i, m_3) && \text{for } w \in \{w_1, w_2, w_3\}; \\
\\
&(m_1, w_i) \succ_c (m_2, w_j) \succ_c (m_3, w_k) && \text{for } c \in \{c_2, c_3\}, \\
&(m_2, w_i) \succ_c (m_3, w_j) \succ_c (m_1, w_k) && \text{for } c = c_1, \\
&(m_i, w_2) \succ_c (m_i, w_1) \succ_c (m_i, w_3) && \text{for } c \in \{c_1, c_2, c_3\},
\end{aligned}
$$

Stable matchings in three-sided systems with cyclic preferences

As a further special case, Boros considers **purely cyclic preferences** orders, where men only care about women, women only care about cats, and cats only care about men. Note that such preferences are incomplete. The concept of purely cyclic preferences naturally extends to the case of s-sided systems. Boros showed that the Gale–Shapley theorem generalizes to the case of purely cyclic preferences provided $s \geq n$, where n is the number of individuals in each group, leaving the case $s < n$ open.

Motivation

Universities can serve as platforms for matching employers and future specialists. Employers support study programs by offering **targeted seats,** where students sign preliminary conditional work contracts. Students have the flexibility to choose between different programs and types of seats. This targeted education approach addresses several coordination problems: students gain new study options with a fast track to employment, employers secure specialized workers early and can influence the curriculum, universities develop stronger programs with better employability outcomes, and society benefits from long-term mutual commitments in the labor market. The key question is: *how can we assist the evolution of this system?*

Setting

The targeted college admission problem is defined by the following components:

- A finite set of students, denoted by $I = \{i_1, \dots, i_n\}$.
- A finite set of schools, denoted by $S = \{s_1, \dots, s_m\}$.
- A finite set of firms, denoted by $F = \{f_1, \dots, f_k\}$.
- Each school $s \in S$ has a capacity q_s, and the vector $q_S = (q_{s_1}, \dots, q_{s_m})$ represents the capacities of all schools.
- Each firm $f \in F$ has a capacity q_f, and the vector $q_F = (q_{f_1}, \dots, q_{f_k})$ represents the capacities of all firms.
- Each student $i \in I$ has an ordinal preference P_i over the set $S \times F$, which represents their preferences over schools and firms.

Setting

- Each school $s \in S$ has an ordinal priority P_s over the power set of students 2^I. The priority is assumed to be responsive, meaning that the school's preferences over groups of students are consistent with its preferences over individual students.
- Each firm $f \in F$ has an ordinal preference P_f over the set $2^{S \times 2^I}$. The preference is also assumed to be responsive. Additionally, it is assumed that for each firm f and school s, the preferences coincide, i.e., $P_f|_s = P_s$.
- The capacity vector $q = (q_S, q_F)$ combines the capacities of schools and firms.
- The preference profile $P = (P_I, P_S, P_F)$ represents the preferences of all students, schools, and firms.

The tuple (I, S, F, q, P) defines the *targeted college admission problem*, which involves matching students to schools and firms while respecting the capacities and preferences of all parties involved.

Plan

- **Existence of a Stable Matching**
  Assume that each school can collaborate with only one firm (while a single firm can collaborate with multiple schools). The firm and the school agree in advance on the number of slots allocated for targeted training. Under these conditions, a stable matching exists. To prove this, we present an algorithm that finds such a stable matching.
- **Prove the Existence of Multiple Stable Matchings**
  Demonstrate that under certain conditions, multiple stable matchings can exist in the model.

Plan

- **Highlight Potential Problems in the Model**
  - **Problem 1: Distribution of Unfilled Slots**
    Initially, it was assumed that all remaining targeted slots would be allocated to budget-funded students. However, it may turn out that a player could benefit from manipulating their preferences to eventually secure a budget-funded spot.
  - **Problem 2: Lack of Monotonicity**
    When a request is rejected (in the case of a firm), it may happen that the firm ends up with no students at all.
- **Provide an Example Demonstrating That a Stable Matching Does Not Always Exist in the General Case**
  Show that, in general, a stable matching may not always exist.

Conclusion

Thank you for your attention!