(4,3)-families of convex sets on a plane

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Overview

- Background and motivation
- Our work

Basic definitions

Definition 1

For a (finite) family of sets $\mathcal{C}$ transversal number of $\mathcal{C}$, denoted by $\tau(\mathcal{C})$, is the minimal cardinality of a set piercing $\mathcal{C}$, i.e. intersecting all the members of $\mathcal{C}$.

Definition 2

Let p and q be positive integers and let $p \geq q$. A finite family $\mathcal{F}$ of convex sets is called satisfying (p, q)-property or just a (p, q)-family if among any p members of $\mathcal{F}$ there are q of them having a point in common.

The (p, q) theorem

(p,q) theorem(Alon, Kleitman, 1992 [1])

Let p, q, d be positive integers such that $p \geq q \geq d + 1$. Then there is a constant c such that for any (p, q)-family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ $\tau(\mathcal{F}) \leq c$

The smallest number c that satisfies conditions of (p, q) theorem is denoted by $HD_d(p, q)$

The problem was first stated in 1957 when in [4] Handwiger and Debrunner prove that if $(d - 1)p < (q - 1)d$, then $HD_d(p, q) = p - q + 1$. So, the "smallest" case when we don't know the exact value of $HD_d(p, q)$ is $(p, q) = (4, 3)$ with $d = 2$

Results on $\mathrm{HD}_2(4,3)$

The Alon-Kleitman proof of (p,q) theorem gives an upper bound on $HD_2(4,3)$ of 343, which is far from optimal, as in [6] it was proved that $3 \leq HD_2(4,3) \leq 13$. In 2020 Daniel McGinnis proved in [7] that $HD_2(4,3) \leq 9$

The proofs of the last two results are pretty complicated although using only some geometric considerations and combinatorial-topological facts such as KKM lemma and Tardos-Kaiser theorem on $d-$intervals[5].

The goal of our work is to find a better upper bound of $HD_2(4,3)$.

The minimal polygon construction

Let $\mathcal{F}$ be a $(4,3)$-family of convex sets on a plane.

The main method of our work is the construction of the minimal(in order of inclusion) compact convex subset U of $\mathbb{R}^2$ such that family $\{A \cap U | A \in \mathcal{F}\}$ still satisfies $(4,3)$-property.

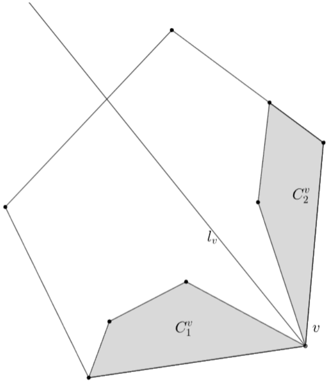
The existence of such set U is corollary of Zorn lemma. Also it's not hard to see that U must be a polygon.

From this momemt we will call such U a minimal polygon of $\mathcal{F}$

The minimal polygon construction

Claim 1

Suppose U is a minimal polygon of a $(4,3)-$family $\mathcal{F}$ of convex sets on plane. Then for any vertex v of U there are two distinct sets C_1^v, C_2^v from $\mathcal{F}$ such that $C_1 \cap C_2 \cap U = \{v\}$



Application to (4,3)-problem

We can also conclude from (4,3) property that for any two vertices v_1, v_2 of U there is a set $C \in \mathcal{F}$ that contains $[v_1, v_2]$. This is particularly interesting when $[v_1, v_2]$ is an edge of U

These observations lead us to the next result:

Claim 2

If U is a minimal polygon of $(4,3)-$family $\mathcal{F}$ and dim $U \leq 1$, then $\tau(\mathcal{F}) \leq 2$

Application to (4,3)-problem

Although we don't have a general solution for the case when $\dim U = 2$, we obtained positive result in some special cases by using ideas from [6] and slightly modifying our construction of U(namely, we first make $\mathcal{F}$ pairwise intersecting and then construct U which also preserves this property of $\mathcal{F}$).

Claim 3

If U is a minimal polygon of $(4,3)-$family $\mathcal{F}$ and there is an edge e of U and a set $C \in \mathcal{F}$ such that $C \cap U = e$ then $\tau(\mathcal{F}) \leq 6$

Application to (4,3)-problem

Also we can conclude the next ,likely new, interesting fact about $(4,3)-$families:

Claim 4

Any (4,3) family of convex sets on a plane can be pierced by a union of one point and one line

This statement can be seen as an improvement of Eckhoff's theorem[3] for $(4,3)-$families. The original Eckhoff's theorem states that if $\mathcal{C}$ is a family of compact convex sets on plane such any 4 sets from $\mathcal{C}$ can be pierced by one line then $\mathcal{C}$ can be pierced by 2 lines.

The plan of future work

It is interesting if we apply some other methods from combinatorics and topology to our problem similarly to how it was done in recent proof of colorful fractional Helly theorem[2].

Specifically, the $(4, 3)$-problem can be easily restated in terms of the nerve complex of $\mathcal{F}$. As nerve complexes of families of convex sets generally have some nice combinatorial features, we assume that we can get better results in this direction.

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