(4,3)-families of convex sets on a plane

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We study the next problem. Let p and q be positive integers and let $p \geq q$. A family $\mathcal{F}$ of convex sets is called satisfying (p, q)-property or just a (p, q)-family if among any p members of $\mathcal{F}$ there are q of them having a point in common. For a (finite) family of sets $\mathcal{C}$ transversal number of $\mathcal{C}$, denoted by $\tau(\mathcal{C})$, is the minimal cardinality of a set piercing $\mathcal{C}$, i.e. intersecting all the members of $\mathcal{C}$.

The (p,q) problem, stated by Hadwiger and Debrunner in [2], asks if for any positive integers p, q, d satisfying $p \geq q \geq d + 1$ there is a number c such that any finite (p, q)-family of convex sets in $\mathbb{R}^d$ can be pierced by c points. A positive answer on this question was given by N.Alon and D.Kleitman in [1]. The smallest number c that satisfies conditions of (p, q) theorem is denoted by $HD_d(p, q)$. In [2] Handiger and Debrunner prove that if $(d - 1)p < (q - 1)d$, then $HD_d(p, q) = p - q + 1$. So, the "smallest" case when we don't know the exact value of $HD_d(p, q)$ is $(p, q) = (4, 3)$ with $d = 2$. The upper bound of $HD_2(4, 3)$ in Alon and Kleitman's proof is 343, which was later improved in [3] by Kleitman, Gyárfás and Tóth who proved that $3 \leq HD_2(4, 3) \leq 13$. The latest improvement in the upper bound is 9 made by D.McGinnis [4]. Our goal is to find better upper bound of $HD_2(4, 3)$ using the construction of the minimal polygon preserving the $(4, 3)$ property of the family.

From now on let $\mathcal{F}$ be a finite $(4, 3)$-family of convex sets on a plane. Consider $\mathcal{U}$ - the set of all compact convex sets U in $\mathbb{R}^2$ such that for any distinct elements A_1, A_2, A_3, A_4 of $\mathcal{F}$ there are three indices $i_1, i_2, i_3 \in [4]$ such that $i_1 < i_2 < i_3$ and $U \cap \bigcap_{j=1}^{3} A_{i_j} \neq \varnothing$. In other words, intersection with U from $\mathcal{U}$ preserves $(4, 3)$-property of $\mathcal{F}$. Note that $\mathcal{U}$ is a partially ordered set, where for two sets $U_1, U_2 \in \mathcal{U}$ $U_1 \leq U_2$ if and only if $U_1 \subseteq U_2$. We prove the next result:

Proposition 1. *There is a minimal element of $\mathcal{U}$, in the sense introduces earlier. Moreover, any such minimal element U is a polygon and for any vertex v of U there are two distinct sets C_1, C_2 from $\mathcal{F}$ such that $C_1 \cap C_2 \cap U = \{v\}$*

We will call such minimal polygon a minimal polygon of family $\mathcal{F}$. Note that if we have a $(4, 3)-$family of pairwise intersecting elements, i.e. the $(2, 2)$-property, we can consider a set of convex compact sets that preserve both $(4, 3)-$

and $(2, 2)$-properties and in that case there also is a minimal polygon with the same features. The next claim allows us to focus on the case when the dimension of minimal polygon of $\mathcal{F}$ is equal to 2:

Proposition 2. *If U is a minimal polygon of $(4, 3)-$family $\mathcal{F}$ and dim $U \leq 1$, then $\tau(\mathcal{F}) \leq 2$*

The general case when $dim(U) = 2$ is rather complicated, but in some special cases we get positive results

Proposition 3. *If U is a minimal polygon of $(4, 3)-$family $\mathcal{F}$ and there is an edge e of U and a set $C \in \mathcal{F}$ such that $C \cap U = e$ then $\tau(\mathcal{F}) \leq 6$*

Also, using the minimal polygon, we prove the next, likely new, result, which may be of independent interest:

Proposition 4. *Any $(4, 3)$-family of convex sets on a plane can be pierced by a union of one line and one point*

References

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