Grading Structure for Derivations of Group Algebras

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Abstract

In this paper we give a way of equipping the derivation algebra of a group algebra with the structure of a graded algebra. The derived group is used as the grading group. For the proof, the identification of the derivation with the characters of the adjoint action groupoid is used. These results also allow us to obtain the analogous structure of a graded algebra for outer derivations. A non-trivial graduation is obtained for all groups that are not perfect.

Calculation of derivations in a group algebra is a well-known problem. Present work elaborates results from articles [1, 2, 3] focused on studying derivations in terms of characters of adjoint action gruppoid. An important result of this research are handy formulas for quick calculation of derivations. These articles explore derivations' link to combinatorial properties of the group.
Among applications note use in coding theory (see [5, 4]), Novikov algebras (see recent work [6]) and more general constructions, like $(\sigma, \tau)-$derivations (see [7]).
Aim of the present work is grading the derivation algebra by identifying derivations and characters on a certain groupoid (all the necessary definitions are given in **??**). The main result of the paper follows.
Let N be a fixed normal subgroup in G such that G/N is abelian.

Theorem 1. *If $|G/N| > 1$, Der is graded with G/N, that is*

$$\mathrm{Der} = \bigoplus_{k \in G/N} \mathrm{Der}_k,$$

$$\forall k, l \in G/N : [\mathrm{Der}_k, \mathrm{Der}_l] \subset \mathrm{Der}_{kl}.$$

Here Der_k is a subalgebra of derivation whose characters' support is localised entirely in one coset $aN = k$. We will give a more precise definition of Der_k in formula ?? with the help of the characters space over a grouppoid of adjoint action.
The most interesting example of grading is the case where N is a derived subgroup.

Moreover, an A-graded algebra can be equipped with the additional structure of dg-algebra, that is there exists for each $a \in A$, such a map $\partial : \mathrm{Der} \to \mathrm{Der}$ that

$$\partial^2 = 0$$

$$\partial(fg) = f\partial(g) + \partial(f)g$$

$$\partial(\mathrm{Der}_k) \subset \mathrm{Der}_{k+a}$$

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