Tree width driven SDP for Max Cut

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Overview

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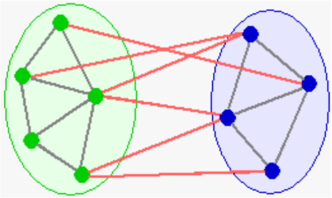
Motivation

- Max Cut problem has applications in many spheres, including machine learning, theoretical computer science, and even theoretical physics.
- It serves as a basis for developing approximation algorithms and heuristic methods for solving other optimization problems.

Problem statement

Let's define W(S) to be the weight of the cut:

$$W(S) = \sum_{i \in S} \sum_{j \notin S} w_{ij}$$



Our goal is to find in polynomial time cut $S_{found} \subseteq V$, such that the value $W(S_{found})$ is as big as possible

$$W(S_{found}) \to \max$$

Problem statement

Let matrix L be L =

$$\begin{bmatrix} (\sum_{i=1}^{n} w_{1i}) & -w_{12} & -w_{13} & \dots & -w_{1n} \\ -w_{21} & (\sum_{i=1}^{n} w_{2i}) & -w_{23} & \dots & -w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & -w_{n3} & \dots & (\sum_{i=1}^{n} w_{ni}) \end{bmatrix}$$

Later we call such matrix the Laplacian of the graph.

Our task is equivalent to finding

$$OPT = \max_{x_i^2=1} x^T L x$$

since

$$\max_{S \subseteq V} W(S) = \frac{1}{4} \max_{x_i^2=1} x^T L x$$

Standard solution

Common plan:
SDP: solve semidefinite programming problem
Decompose solution X with Cholesky decomposition
Round the solution by introducing random hyperplane
Get the approximate Max Cut solution
The quality of approximation will be at least 0.878 of optimum.

$$OPT = \max_{x_i^2=1} x^T L x \leq \max_{\substack{X \succeq 0 \\ diag(X)=1_n}} (LX) = SDP$$

Our goal

If Unique Games Conjecture is true, it is known to be impossible to find solution, which is asymptotically better, than the one with constant 0.878...

Our goal is to develop a non-asymptotic improvement of the solution in polynomial time on arbitrary graphs by applying tree-width approach to the methods for solving SDP.

Current progress

$$OPT = \max_{x_i^2=1} x^T L x = \max_{\substack{x_i^2=1 \\ X=x^T x}} LX = \max_{\substack{X \succeq 0 \\ diag(X)=1_n \\ rank(X)=1}} (LX)$$

Let's find the dual for OPT problem.

$$Dual = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i (1 - x_i^2) - \sum_{i,j}^{n} x_i x_j L_{ij} =$$

$$= \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2$$

Current progress

if $-L - Diag(\lambda) \not\succeq 0$ then

$$\min_x \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2 = -\infty$$

since we can multiply the vector, which proves that $-L - Diag(\lambda) \not\succeq 0$, by constant and get the arbitrary small value.
And if $-L - Diag(\lambda) \succeq 0$, then

$$\min_x \sum_{i=1}^{n} \lambda_i - \sum_{i,j}^{n} x_i x_j L_{ij} - \sum_{i=1}^{n} \lambda_i x_i^2 = \min_x \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i$$

Current progress

This means that if we denote $\xi_i := -\lambda_i$, Dual problem can be rewritten this way:

$$Dual = \max_{\lambda} \min_{x} \sum_{i=1}^{n} \lambda_i (1 - x_i^2) - \sum_{i,j}^{n} x_i x_j L_{ij} = \max_{\substack{\lambda: \\ -L - Diag(\lambda) \succeq 0}} \sum_{i=1}^{n} \lambda_i =$$

$$\max_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} -\xi_i = \min_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} \xi_i$$

Current progress

Lemma

$$Dual = \min_{\substack{\xi: \\ Diag(\xi) \succeq L}} \sum_{i=1}^{n} \xi_i = \min_{L_T \succeq L} \max_{x: x_i^2 = 1} x^T L_T x = TreeRel$$

where L_T can be represented as $L_T = L_{tree} + Diagonal$, where L_{tree} corresponds to Laplacian of a tree graph and Diag is a diagonal matrix with non-negative values.

Current progress

$$H_k = \min_{\substack{T : T = T^\top \succeq A \\ \mathrm{tw}(T) \leq k}} \max x^\top T x, \qquad OPT = H_k \leq \cdots \leq H_1 = SDP$$

where optimization is taken over all graph with tree-width less than k. That is internal problem can be solved by dynamic programming.
Plans:
Expand Dual problem to problem with 3/5/7-diagonals and solve it
Having T, optimal approximation with $\mathrm{tw}(T) = 1$ (tree), incrementally add edges of the graph with the largest weight to T keeping $\mathrm{tw}(T) \leq k$.