История и философия производной

Симфония из определений и свойств

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Recall that **Group algebra** $\mathbb{C}[G]$ is an algebra of formal finite sums of type $(a_1, \ldots, a_n \in \mathbb{C}, g_1, \ldots, g_n \in G)$

$$a_1 g_1 + \cdots + a_n g_n$$

The (contemporary) Moments of flowing quantities are as the Velocities of flowing or increaſing; that is, as their Fluxions. Now if this be proved of Lines, it will equally obtain in all flowing quantities whatever, which may always be adequately repreſented and expounded by Lines. But in equable Motions, the Times being given, the Spaces deſcribed will be as the Velocities of Deſcription, as is known in Mechanicks. And if this be true of any finite Spaces whatever, or of all Spaces in general, it muſt alſo obtain in infinitely little Spaces, which we call Moments. And even in Motions continually accelerated or retarded, the Motions in infinitely little Spaces, or Moments, muſt degenerate into equability. So that the Velocities of increaſe or decreaſe, or the Fluxions, will be always as the contemporary Moments. Therefore the Ratio of the Fluxions of Quantities, and the Ratio of their contemporary Moments, will always be the ſame, and may be uſed promiſcuouſly for each other.

Sir Isaac Newton. The method of fluxions and infinite series

People have very different ways of understanding particular pieces of mathematics. To illustrate this, it is best to take an example that practicing mathematicians understand in multiple ways, but that we see our students struggling with. The derivative of a function fits well. The derivative can be thought of as:

(1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.
(2) Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, the derivative of $f \circ g$ is $f' \circ g * g'$, *etc.*
(3) Logical: $f'(x) = d$ if and only if for every ϵ there is a δ such that when $0 < |\Delta x| < \delta$,

$$\left|\frac{f(x + \Delta x) - f(x)}{\Delta x} - d\right| < \delta.$$

(4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.
(5) Rate: the instantaneous speed of $f(t)$, when t is time.
(6) Approximation: The derivative of a function is the best linear approximation to the function near a point.
(7) Microscopic: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.

This is a list of different ways of *thinking about* or *conceiving of* the derivative, rather than a list of different *logical definitions*. Unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions.

I can remember absorbing each of these concepts as something new and interesting, and spending a good deal of mental time and effort digesting and practicing with each, reconciling it with the others. I also remember coming back to revisit these different concepts later with added meaning and understanding.

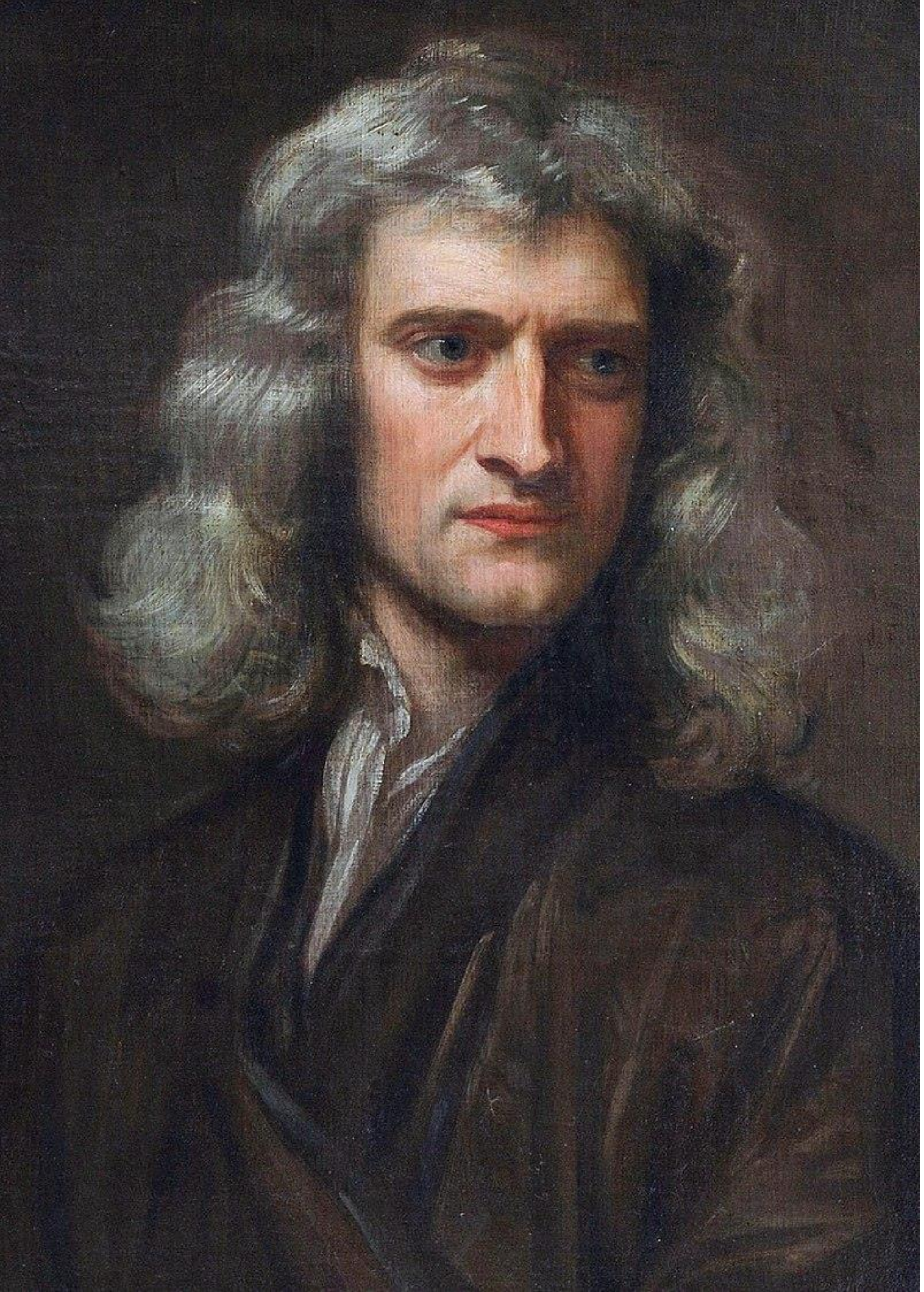
$$d(y(x)) = y'dx$$

$$d(fg) = df\, g + f\, dg$$

$$d(y(x(t))) = y'_x d(x(t))$$

https://mathoverflow.net/a/108804/161380

“…и быстрых разумом Невтонов…” М.В. Ломоносов



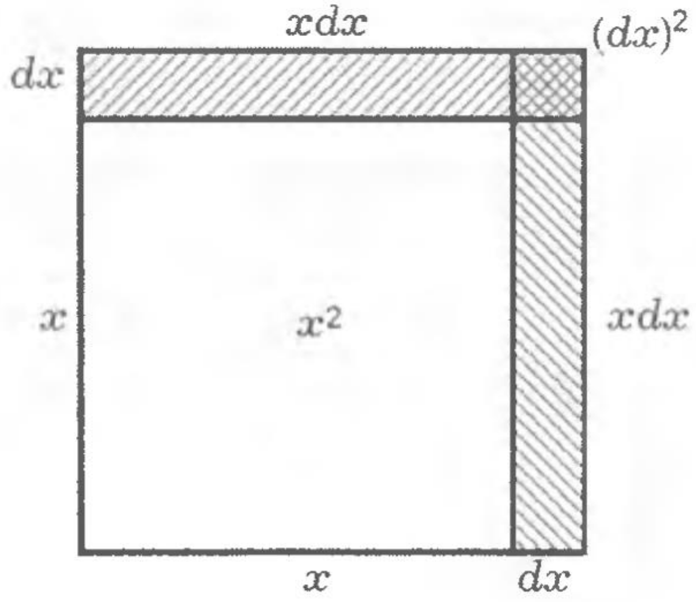


Рис. 1. Экспериментальное опровержение Ньютоном «правила Лейбница».

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We define **derivation** as a linear operator d that satisfies the Leibniz rule (for all $a, b \in \mathbb{C}[G]$)

$$d(ab) = d(a) \cdot b + a \cdot d(b)$$

Интересные ссылки

- Newton. *Method of Fluxions* at the Internet Archive https://archive.org/details/methodoffluxions00newt/page/252/mode/2up
- В.И. Арнольд.
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- https://mathoverflow.net/q/117374/161380. Why is the Leibniz rule a definition for derivations?