Helly-type problems and families of convex sets satisfying (p, q)-property

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Overview

1. Classic results
   - Helly's Theorem
   - Generalizations of Helly's theorem
2. Extension of colorful Helly Theorem
3. Our problem and current progress

Helly's Theorem

Helly's theorem is one of the cornerstones of combinatorial convexity, providing an effective method for determining the intersection pattern of a finite family of convex sets

Theorem 1 (Helly's Theorem)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every d + 1 or fewer sets of them have a non-empty intersection, then $\bigcap \mathcal{F} \neq \varnothing$, where $\bigcap \mathcal{F} := \bigcap_{C \in \mathcal{F}} C$

Theorem 2 (fractional Helly)

Let $\alpha \in (0,1]$ and $d \geq 2$ be fixed. If $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, $|\mathcal{F}| = n$, with at least $\alpha\binom{n}{d+1}$ intersecting d + 1 tuples, then there exists an intersecting subfamily $\mathcal{F}' \subset \mathcal{F}$, with $|\mathcal{F}'| \geq \beta n$, where $\beta > 0$ is a constant that depends only on d and α.

Theorem 3 (colorful Helly)

Let $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap\limits_{i=1}^{d+1} C_i \neq \varnothing$ for all $C_1 \in \mathcal{F}_1, ..., C_{d+1} \in \mathcal{F}_{d+1}$ then there exists $i \in [d+1]$ such that $\bigcap \mathcal{F}_i \neq \varnothing$

Extension of colorful Helly Theorem

The colorful Helly theorem only provides information that some family has a non-empty intersection, and we do not know anything else about this intersection. Next result provides some additional information on it

Theorem 4 (W.Rao)

Let $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap\limits_{i=1}^{d+1} C_i \neq \varnothing$ for all $C_1 \in \mathcal{F}_1, ..., C_{d+1} \in \mathcal{F}_{d+1}$ then $\sum\limits_{i=1}^{d+1} dim \bigcap \mathcal{F}_i \geq 0$, where $dim\ \varnothing = -1$

Method of Proof of Theorem 4

The proof of Theorem 5 uses reduction to polytopes and considers minimal with respect to inclusion compact convex set U such that families $\mathcal{F}_1 \bigcap U, ..., \mathcal{F}_{d+1} \bigcap U$ satisfy colorful Helly property.

It is easy to see that U is a polytope and every vertex of U is contained in the intersection of some family $\bigcap \mathcal{F}_i$. Then the statement of the theorem is proved by induction on d.

The Problem

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. For $p \geq q \geq d+1$ we say that $\mathcal{F}$ has (p, q) property if among any p sets from $\mathcal{F}$ there are q having a common point

It is well known that for any family $\mathcal{F}$ in $\mathbb{R}^d$ satisfying the (p, q) property there is a set of fixed size $HD_d(p, q)$ that intersects all the sets in $\mathcal{F}$. However, the minimal value of $HD_d(p, q)$ is unknown even for $HD_2(4, 3)$. It is only known that $HD_2(4, 3) \geq 3$ and $HD_2(4, 3) \leq 9$ (Daniel MCGinnis)

The Problem

It is interesting to see if we can apply method of proof of Theorem 4 McGinnis's paper to get a better upper bound on $HD_d(p, q)$ and, as the easiest case, $HD_2(4, 3)$

Current progress

Following the method of proof of Theorem 4, consider family $\mathcal{U}$ of all compact convex sets U such that $\mathcal{F} \bigcap U$ has (4,3)-property.

It is easy to check that $\mathcal{U}$ has minimal element U with respect to inclusion and that U is a polytope.

Then it was proved that for every vertex v of U there are two sets C_1, C_2 from $\mathcal{F}$ such that $C_1 \cap C_2 \cap U = \{v\}$. Using this fact it can be shown that if $dimU < 2$ $\mathcal{F}$ can be pierced by 2 points.

Throughout the next part of presentation we fix on the case when $dimU = 2$

Current progress

Let v be a fixed vertex of U and C_1, C_2 be the sets from F such that $C_1 \cap C_2 \cap U = \{v\}$.

Note that by separation theorem there is a line l_v such that $C_i \cap U \cap l_v = \{v\}$

Also we can conclude from (4,3) property that for any two vertices v_1, v_2 of U there is a set $C \in \mathcal{F}$ that contains $[v_1, v_2]$. This is particularly interesting when $[v_1, v_2]$ is an edge of U

Current progress

Claim

There is a point $p \in \mathbb{R}^2$ such that all the sets from $\mathcal{F}$ that don't contain p are pairwise intersecting

Proof: Let G be the graph whose vertices are the sets from $\mathcal{F}$ and two set A and B form an edge iff they are disjoint.

Note that by (4,3) property there are no disjoint edges and triangles in this graph. Thus, the only connected component with edges of G consists of the vertices $A, B_1, \dots B_n$ and edges $(A, B_1), \dots, (A, B_n)$.

Again, by (4,3) property we have that any three of the sets $B_1, \dots, B_n$ have a common point. Then, by Helly's theorem we have that all the sets $B_1, \dots, B_n$ have a common point, which finishes the proof.

Note that we can construct the minimal polytope U for a pairwise intersecting family $\mathcal{F}$ that will conserve all pairs of sets intersecting and (4,3) property of $\mathcal{F}$ and it will have the same features as the minimal polytope intodNotices before.

Thus, we obtain the next interesting fact:

Claim

Any (4,3) family of convex sets on a plane can be pierced by a union of one point and one line

Proof: just take point p from previous claim and a line l_v for any vertex v of U

Current progress

The next theorem was proved by D.Kleitman, A.Gyarfas and G.Toth in [2]:

Theorem(Kleitman, Gyarfas, Toth)

Let Q be a "horizontal"line and $\mathcal{F}$ be a family of convex polygons such that any two polygons from $\mathcal{F}$ meet on or above Q, and if three sets from $\mathcal{F}$ are mutually disjoint on Q then they have a common point above Q. Then $\mathcal{F}$ can be pierced by 5 points.

From this theorem we can deduce the solution of our problem in some easy cases:

Claim

If there is a set C from $\mathcal{F}$ such that $C \cap U$ is an edge of U then $\mathcal{F}$ can be pierced by 6 points

KKM lemma

Let Δ^n be n-dimensional simplex in $\mathbb{R}^{n+1}$ with vertices $e_1, \ldots, e_{n+1}$, where $e_1, \ldots, e_{n+1}$ form an orthonormal basis of $\mathbb{R}^{n+1}$

Theorem(KKM lemma)

Let Δ^n be n-dimensional simplex and $A_1, \ldots, A_{n+1}$ are open(closed) subsets of Δ^n. If for each face σ of Δ^n $\sigma \subset \bigcup_{e_i \in \sigma} A_i$ then $\bigcap A_i \neq \varnothing$

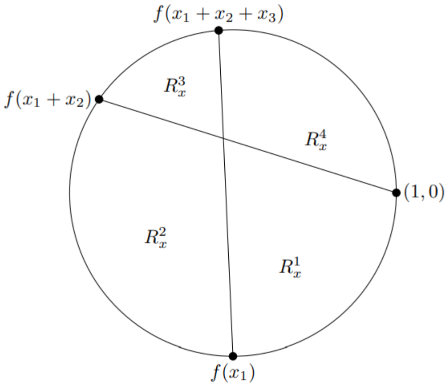
For a family $\mathcal{F}$ of sets $\nu(\mathcal{F})$ is the maximal cardinality of a subset of $\mathcal{F}$ consisting of pairwise disjoint sets, and $\tau(\mathcal{F})$ is the minimal cardinality of a set intersecting all the members of $\mathcal{F}$.

d-interval is a union of intervals lying on d disjoint lines, which containing one interval per a line.

Theorem(Tardos, Kaiser)

Let $\mathcal{F}$ be a finite family of d-intervals. If $d = 1$ then $\tau(\mathcal{F}) = \nu(\mathcal{F})$. If $d > 1$ then $\tau(\mathcal{F}) \leq (d^2 - d)\nu(\mathcal{F})$

Using KKM theorem



We can divide plane by two intersecting lines depending on $x \in \Delta^3$ into four parts R^i_x, $i \in [4]$. Using this division we can define an open covering A_i of Δ^3.

Using KKM theorem

Theorem(McGinnis)

If $\bigcup A_i \neq \Delta^3$ then $\tau(\mathcal{F}) \leq 8$

Now fix $x \in \bigcap A_i$ and two lines corresponding to it. Sets of $\mathcal{F}$ not containing the point of intersection of these two lines can be divided into four parts $\mathcal{C}_1, ..., \mathcal{C}_4$, where sets from $\mathcal{C}_i$ don't intersect R_x^i.

Theorem(McGinnis)

For all $i \in [4]$ $\tau(\mathcal{C}_i) \leq 2$

Our goal

Using our minimal polygon we can make a slight improvement in this construction:

Claim

There is $x \in \bigcap A_i$ such that all sets from $\mathcal{F}$ intersect $R_x^i \cup R_x^{i+1}$ for some i. Moreover, for some $j \in i, i+1$ all pairs of sets from $\mathcal{C}_j$ have a common point in $\overline{R_x^{i+2} \cup R_x^{i+3}}$

Globally, our goal is to show that estimations $\tau(\mathcal{C}_i) \leq 2$ cannot be all sharp at the same time

References

[1] Daniel McGinnis "A family of convex sets in the plane satisfying the (4,3)-property can be pierced by nine points"https://arxiv.org/abs/2010.13195
[2]D. J. Kleitman, A. Gyarfas, and G. Toth. Convex sets in the plane with three of every four meeting. Combinatorica, 21:221–232, 2001.